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Soledad Estrella

THE TABLE OBJECT:
AN EPISTEMOLOGICAL, COGNITIVE AND DIDACTIC STUDY

Thesis directed by

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CHILE

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INTRODUCTION

This dissertation focuses on the table as a learning goal in the first years of schooling.

The author, based on professional motivation and experience, addresses the difficulties in learning and teaching tables by first entering into a historical epistemological study of the issue, a substantial task in the Pontifical Catholic University of Valparaiso Doctorate Program in Didactics of Mathematics, and is able to unravel the constituent elements of the table as a rectangular network and its social uses as regards types of interpretations. Additionally, the author examines the status of the table in a school context according to recollection and exploratory analysis of data.

The dissertation then establishes the context of demands that society currently places on the curriculum, and in this context, based on currently dominant tendencies in statistics education, provides guidelines for dealing with tables in the classroom and maintaining high cognitive demands during the process -Stein and Schmidt-, characterizing in detail the underlying semiotic representations and theoretical conceptual structures -Vergnaud- that emerge in the process of conceptualization of the table in the first years of school. The learning method for statistical reasoning -Garfield and Ben-Zvi-, and in particular *transnumeration* -Wild and Pfannkuch- emerge as paradigms that accommodate the teaching proposal, meeting the social demand for curricular renovation and the international tendency to exceed the national status quo of arithmetization in statistics education.

This dissertation makes evident the *paramathematical* status of tables in the national curriculum, which sets tables aside as a tool and does not consider them a teaching object. The dissertation also elucidates the cognitive demands that studying tables makes on students, exploring in light of this an *ad hoc* table taxonomy and a proposal for critical education aimed at developing competence in representation and analysis of data for decision making.

SUMMARY

Chapter one presents the author's interests in statistical education and the research problem. It also presents the research questions and their goals.

Chapter two presents a review of the specialized literature pertinent to tables in statistics, focused on the learning difficulties that the table format presents and the implications for teaching. Additionally, it introduces the theoretical framework for the cognitive and didactic aspects of the research.

Chapter three offers a panoramic vision of the historic process of evolution of ideas about tables, their connotations as a human tool, and their emergence and development in diverse cultures and various moments in history, issues that contribute to understanding this object and its didactic reach. This chapter shows us the table and its presence as a storage tool, a calculation tool in numbering and meteorological systems, an analysis tool in scientific and mathematical fields, and its relation with the creation of numbers and the concept of a function.

Chapter four provides the scholastic status of tables. It addresses an epistemological goal regarding tables as a significant element in the analysis of the circulation of knowledge and its normalization. Going into greater depth in characteristics of the table format based on information science and statistics, a generic model for the table is proposed. This chapter investigates cognitive aspects, as it studies tables as representations that support the construction of meanings for data and identifies subjects' roles and cognitive processes associated with statistical tables. The chapter ends with a study of the role of tables in the list items of an international primary school test and its status in the current curriculum in Chile and three other countries.

Chapter five presents four studies that give us an approximation of the understanding of table learning at school level. It begins with an analysis of the evidence that emerges from student productions and instructor management in task demands, in a data analysis situation. It continues with the characterization of the types of interpretation that dealing with tables

demands, proposing categories to create a hierarchy of understanding specific to tables, which finally will be tested.

Chapter six completes the work of the dissertation with the conclusions and findings for a first exploration of the progressive dominance of the conceptualization of tables by students in the first years of school, supported by the theoretical references and collected results.

In order to offer the reader a panoramic view, each chapter begins with an ordered list of the contents that we have titled “Chapter Summary”.

CHAPTER I

Study Foundations

Former les citoyens à la pensée de la variabilité et à la gestion de l'aléatoire n'est pas seulement, aujourd'hui, une question socialement vive : c'est aussi une question didactiquement vive.

Chevallard and Wozniak (2006)

Chapter Summary

1. INTRODUCTION

This chapter addresses the focus, the central issues, and the questions that delimit the evolution of the study, and ends with the dissertation's central questions and goals.

This dissertation enters in the general framework of human cognition and in the making sense of a semiotic representation: the table.

The dissertation focuses on statistics education at the school level. More precisely, it is dedicated to elucidating the cognitive demands that tables impose on the learner and determining the cognitive demands to which the teacher recurs to orchestrate the learning of data analysis.

The research is carried out at the primary school level, where paper and pencil techniques are still prevalent. The research is centered on the data, at the heart of statistics, in order to analyze the data and describe its behavior, obtain information, and make reasoned decisions.

2. ISSUES THIS DISSERTATION ADDRESSES

Social demands for including statistics in schools

The insertion of statistics in schools has forced a dynamic upon the institutions that has not been addressed positively or successfully in Chile, an issue that complicates the work of a teacher educated for mathematics teaching and learning, upon whom the curriculum imposes a teaching task for which he or she has no experience or training. As such, statistics is included from the first years and throughout schooling, making it a difficult to address challenge for teachers who must produce this learning in their students, and a source of didactic phenomena.

Today in Chile, statistics education is considered part of learning and debate in society; it is present in the media and in schools.

In this respect, Chile is in step with declarations by various international organisms. For example, the *World Conference on Science for the 21st Century*, organized by UNESCO and by the International Council for Science (ICSU[1]), in its *Declaration on Science and the Use of Scientific Knowledge*, made in Budapest in 1999, expresses that, "... access to scientific knowledge for peaceful purposes from a very early age is part of the right to education belonging to all men and women, and that science education is essential for human development, for creating endogenous scientific capacity and for having active and informed citizens." It adds: "Science education, in the broad sense, without discrimination and encompassing all levels and modalities, is a fundamental prerequisite for democracy and for ensuring sustainable development." From 2005 to 2008 in Europe, a project was developed to improve the early teaching and learning of statistical reasoning in European primary and secondary schools[2]. Later, the National Council of Teachers of Mathematics of the United States identified data analysis and probability as its "Focus of the Year" for the period 2007-2008.

In 2010, the United Nations (UN) established World Statistics Day in order to make known the importance of this discipline in many decisions made by governments, businesses, and communities. Statistics participates in the planning of schools, hospitals, roads, and much more, and delivers information and understanding regarding tendencies and forces that affect daily life.

What's more, researchers in statistical education have displayed their certainty that educating citizens in thinking about variability and dealing with randomness is not only a social issue, but also a didactic issue, as expressed by Chevallard and Wozniak (2006).

The majority of the world's school curricula have recently incorporated a theme of statistics and probability in mathematics courses from the first years of schooling. This new content creates tension for teachers in their work and training, as their professional curriculum does not include solid proposals regarding statistics and statistics education, and there is no tradition of school-level statistics education. Additionally, teachers are not familiar with data

[1] *International Council for Science*, formally *International Council of Scientific Unions*, whose initials have been maintained

[2] *Enhancing the Teaching and Learning of Early Statistical Reasoning in European Schools* project (2009, http://www.earlystatistics.net/)

analysis, with variability in data analysis, nor with non-determinism in mathematics (Batanero, Burrill, and Reading, 2001).

According to available information, the relationship between mathematics and statistics at the school level is not clear to teachers; in fact, it does not appear that, in general, they question this relationship, and a certain arithmetic reduction of statistics can be found in classrooms, which acts as "seeing" the data as numbers, and not as numbers with a context (Cobb & Moore, 1997).

3. Current status of tables in the Chilean curriculum

The table as a representation used in mathematics and statistics cuts across the entire curriculum. It appears as a content item in the statistics course, and also, explicitly, in all the themes of the mathematics course. With different functionalities and appearances, with a long history of everyday use, the table appears transparently in classrooms and textbooks. As a *paramathematical* object, tables do not make up a teaching item, but rather an “auxiliary" knowledge object, which must be "learned" (or which one must "get to know"), but which is not "taught" although it is necessary for teaching (and learning) the knowledge objects (Chevallard, 1985, 1997).

There are known conventions regarding making tables, but these are not taught.

Pfannkuch and Rubick (2002) point out that there is little research on students and the construction and interpretation of tables of statistical data and that research is needed on how students perceive them. Existing research supports that tables should be created by students (Cox 1999). However, when teaching tables, they are usually given to students already made. What's more, the results of this type of research do not appear to reach curriculum designers or textbook designers (Shaughnessy, 2007).

Nisbet *et al.* (2003) find that the process of reorganizing numerical data in frequencies is not an intuitive process for children in grades 1 to 3; Pfannkuch and Rubick (*op. cit.*) observe that tabulating data is an ability that requires determining the way to present it clearly and unambiguously, which implies some loss of the data's information, and conclude in their analysis that dealing with data in this way is an ability "more sophisticated than we had thought" (p. 13).

Statistical tables are an explicit curriculum content, but the curriculum is concerned with "doing", and the table is considered a transitional tool, in a state of “tabular technique”. As such, as a tool, we can easily recognize the pragmatic value of tables, but it appears to be more difficult to determine their epistemological value, that is, their role as a mathematical object. Putting the epistemological value of tables in its place in the didactic system requires reflection and rebuilding, in which pertinent data analysis situations promote the emergence of their role as a tool (techniques) and as an object (concepts) and give tables status in the institution in order to contribute to learning tables and the objects involved with them.

This dissertation seeks to reveal tables in light of the complexity of the representations in which they intervene, their invisibility in teaching, their epistemological status as a tool and object, and their functionality in school statistics.

4. Issues that define the evolution of the dissertation

Expertise of the author that enriches the study

The author’s academic training and experience in research projects related to statistics have allowed her to enrich her educational reflections and enter into greater complexity regarding didactics of statistics.

Regarding teacher training, this education and experience includes: elaboration, with other academics belonging to the Chilean Statistics Society (SOCHE), of a public document presented to the Chilean Ministry of Education, titled “*Recommendations for School Curriculum in the Area of Statistics and Probability*”, which was well received; collaboration in the evaluation of the “Standards for Education in Mathematics for Middle-School Teachers” in the area of Statistics and Probability, carried out in the framework of a project of the University of Chile Center for Advanced Research in Education (CIAE); participation in two phases of the creation of items for the INICIA evaluative instrument, which forms part of the

INICIA Program[3], administered by the Ministry of Education for recently graduated professionals; participation in initial and continued training at the university level in didactics of statistics. This has allowed her to verify difficulties of teachers as well as students in forming concepts and representations in statistics and probability.

At the same time, her experience as a primary and secondary school teacher, and in programs for academically talented students, have allowed her to go more in depth in the issues related to statistics learning and teaching for school students.

Additionally, in her Master's Thesis in Didactics, the author developed an instrument for evaluating the *pedagogical content knowledge* in statistics of basic education professors, which was the statistics component of the Ministry of Education's Fund for Research and Development in Education (FONIDE) project 10980[4], which provided empirical evidence of difficulties of children and adults in understanding an item related to tables. This finding and a revision of prior representations of data allowed the author to enter in greater depth in the difficulties and errors of the adults and children in statistical tasks related to understanding graphs and tables, as well as in the epistemological obstacles to the knowledge in question.

Potentialities of the table for dealing with data

Statistical literacy includes basic important abilities that can be used to understand statistical information or research results. These abilities include the capacity to organize data, create and visualize tables, and work with different representations of data.

Among the representations of data in statistics, tables and graphs stand out, tables specifically in their roles of organizational support and analytical tool.

[3] The INICIA Program is a component of the Program for Promotion of Quality in Initial Teacher Training. It is an initiative of the Chilean Ministry of Education whose goal is to the strengthen teacher training provided by the country's institutions of higher learning. Its strategic lines are: defining orientations for initial teacher training and standards for each pedagogy major; design and implementation of a diagnostic evaluation of the knowledge of graduates with pedagogy majors.

[4] Project of FONIDE IV, "Pedagogical Content Knowledge and its Incidence in Mathematics Teaching at the Basic Education Level" by researchers Raimundo Olfos and Ismenia Guzman.

Tables, by registering all the data, aim at the specific and allow for displaying data that have different units of measurement. In tables' comparative function, the relations that present themselves interact principally with language (verbal and symbolic) and only secondarily with visual aspects, as their primordial function is not to show tendencies, but rather to deliver all of the data, capturing the totality of the "reality" they register.
An advantage of tables is that they are less susceptible to the external manipulations and simplifications that graphs can suffer; their disadvantage is that, for a large quantity of data, their efficacy in visually displaying the behavior of the data diminishes.

Exploratory Data Analysis (EDA) (Tukey, 1977) gives great importance to thinking, conjecturing, and learning about data before and during the construction of visual representations and promotes and values the use of graphical and tabular representations as a good analysis tool, not only as a way of registering data. A fundamental idea of EDA is that the use of diverse and multiple representations of data has a potentially positive role in the development of knew knowledge and intuitions. Examples of these activities are converting data into tables, tables into graphs, lists of numbers into representations like stem-and-leaf plots, and making graphs that allow for comparing among various samples.

The definition of a table and its use in school

Table is a polysemic word, and as an object it presents distinct and interesting characteristics and functionalities to study. Tables can be considered physically as a segmented rectangular network, whose cells contain headings and values arranged logically, and whose function lies in concentrating semantic relations.

As school content, the table is present in practically every course; its cells can contain only images, only text, or only numbers, or a combination of these. The different contents and their placement create different readings: a table of verb tenses differs from a table of historic events, or from the periodic table of the elements, or from multiplication tables, or truth or proportionality tables.

As content to be taught, tables are found in the mathematics program, specifically in statistics, principally as data and frequency tables.

As mentioned earlier, tables are used as a tool, but avoided as an object to be taught. The Chilean national curriculum, in the area of statistics (data treatment or data and probabilities) in the mathematics course, demands that students be capable of collecting, registering, analyzing, comparing, representing, reading, interpreting, and completing tables.

In order to create a table, a set of graphical requirements and specific cognitive capacities are needed. Some of the graphical requirements are situating categories in two dimensions, horizontal and vertical, creating cells, defining category labels, and writing the results of external or internal measurements in the cells. Some of the specific implicit cognitive abilities are categorizing subjects based on a variable, and, eventually, categorizing subjects based on a second variable and cross-categorizing the subjects according to both variables, and, in that case, counting the elements of each crossed category (frequencies).

5. RESULTS OF RESEARCH ON STATISTICAL REPRESENTATIONS

A revision of the statistics education research literature from 1995 to 2005 was published by Garfield and Ben-Zvi (2007); it offers a general panorama of studies on teaching and learning statistics at all levels. In particular, it points out that the importance and the complexity of statistical representations are found in lines of work such as: development of reasoning about distribution, center, and variability; development of understanding of statistics by teachers in their initial and continued training; errors and erroneous conceptions in reasoning about statistics and probability; visualization and exploration of data with technological tools.

Shaughnessy (2007), in the United States *National Council of Teachers of Mathematics* (NCTM) “Handbook of Research on Mathematics Teaching and Learning” sustains that lines of research on conceptual issues and teaching issues in statistics are required. He points to a national study of statistics items, applied from 2000 to 2003, whose results show poor performances (by U.S. students) on items that involve interpretation or application of information in tables or graphs. Among other guidelines, the Handbook proposes research on proposals of high cognitive level data analysis tasks that promote critical analysis and multiple representations in the classroom; in statistical literacy, he proposes investigating testing of reading and critical evaluation of information in tables and graphs.

Studies of tables and graphs in the context of statistical literacy

A book sponsored by the International Commission on Mathematical Instruction (ICMI) in collaboration with the International Association for Statistical Education (IASE) in 2011, collects research from around the world on statistics teaching in schools and challenges in teaching and teacher training. Although it centers on education and professional development for teachers to teach statistics, and the elements necessary for this education at the school level, it also includes research on reasoning and learning in teachers and students. Also, considering that one of the components of statistical literacy is familiarity with the ideas that graphic and tabular representations include, it includes research that deals with this specifically, for example: MacGillivray and Pereira-Mendoza on teaching statistical thinking (Batanero, Burrill, and Reading, Eds., 2011), teachers' graphical competence, by Gonzalez, Espinel, and Ainley (ibid.), and "teaching to teach" research statistics from Makar and Fielding-Wells (ibid.).

There are various studies on understanding of graphs and/or tables, among them Curcio (1989); Friel, Curcio, and Bright (2001); Aoyama (2007); Espinel (2007); Tauber (2006); Arteaga, Batanero, Cañadas, and Contreras (2011); Ben-Zvi and Arcavi (2001); Ben-Zvi and Sharett-Amir (2005), as well as studies based on characterizing statistical literacy, such as Gal (2002, 2004); Ben-Zvi and Garfield (2004); Schield (2000, 2006); Shaughnessy (2007); Watson (2006); and Burrill and Biehler (2011).

An attribute inherent to representations is *transnumeration*, a process that refers to the elements of thought involved in understanding information related to different representations of data in diverse modalities (tables, calculations of statistical summaries, graphs, etc.).

This term, which identifies the process of "changing representations to create understanding" (Wild and Pfannkuch, 1999, p. 227), addresses the fact that sometimes in exploring data, a representation makes something new and unknown manifest, creating greater understanding of the problem. Shaughnessy (2007) points out that, by any means, *transnumeration* is a term that needs greater precision, and that a culture of learning and teaching that stimulates this could evolve if professors and curriculum developers consider the recommendations supported by research that indicates the need for students to have more opportunities to create their own representations of data rather than work with ready-made tables and graphs.

Status of the understanding of tabular representations

As we will see later, in the process of understanding a table, various processes must be activated, among these, reading, search and interpretation, and evaluation. The process of producing a table implies processes of writing, of creating, and of completing. Putting data in its place activates understanding of the relational structure of the table and opposes a reading with simplified language -about the variable and its categories- due to the eventually enormous quantity of information reduced to a bi-dimensional space.

The processes of comprehension vary in complexity according to whether one is dealing with a table with a data registry, a transition table for making calculations, or a table that allows for analysis and ends up being a synthesis tool in exploratory data analysis. A comprehensive reading of a statistical table includes considering its context and looking for the behavior of the data.

Pfannkuch and Rubick (2002, p.5) identified specific instances of transnumeration in statistical thinking: (1) making measurements that capture the characteristics of the real situation; (2) transforming the initial data in other representations -such as ordered data, graphs, tables, and statistical summaries- to look for meaning in the data; and (3) communicating to others in terms of the meaning of the real situation.

6. ISSUES RAISED BY THE INCLUSION OF STATISTICS IN THE CURRICULUM

The recent inclusion of statistics in the school curriculum poses new issues for research from the beginning of schooling. The familiarity that tables present has derived in a certain transparence of the table, present in schools as a tool, but not as a learning object. In an area as important as EDA, statistical representations like tables take on relevance, as they allow for discovering characteristics in data that facilitate resolving the problem in question.

A review of the background of current research in statistics confirms the necessity of addressing one of the components of statistical literacy, representations, and, in particular, frequency tables at the primary school level, as an analysis tool in data exploration and also as an object that is defined, that needs to be examined, and with which operations appropriate to its structure can be carried out.

The functional diversity of the table and its cross-cutting presence in school subjects and in real life demand that we look at how it is learned and taught.

As a didactic problem in statistical education, it is interesting to research the cognitive processes of learners faced with tabular representations and the direction of the teacher in a lesson with statistical representations. Both points of the process require empirical investigation in school environments of teaching and learning statistical tables.

{0>La revisión de literatura muestra que, en los estudios sobre enseñanza y aprendizaje de la Estadística referentes a los primeros años de escolaridad, las preguntas de investigación que se plantean son, principalmente:<}0{>A literature review shows that in studies of teaching and learning statistics in the first years of schooling, the research questions proposed are, principally:<0} {0>¿Cuáles son algunos de los errores y malinterpretaciones en el razonamiento de Estadística?<}0{>What are some of the errors and misconceptions<1} in statistical reasoning?{1><1}<0} {0>¿Cómo llegan los escolares a comprender la Estadística?<}0{>How do school children come to understand statistics?<0} {0>¿Cómo desarrollan los alumnos de pedagogía y los profesores en servicio la comprensión de la Estadística?<}0{>How do pedagogy students and service teachers develop understanding of statistics?<0} {0>¿Cómo inician los profesores el reto de *alfabetizar estadísticamente* a sus estudiantes?<}0{>How do teachers begin the challenge of *creating statistical literacy* in their students?<0} {0>Garfield y Ben-Zvi (2007) señalan que los estudios del área revisados en su conjunto muestran las dificultades que tienen los estudiantes con el aprendizaje de la Estadística y la necesidad de revisar los métodos de enseñanza.<}0{>Garfield and Ben-Zvi (2007) point out that studies in the area, reviewed as a whole, show the difficulties students have in learning statistics and the necessity of revising teaching methods.<0}

7. Research questions

This dissertation enters in a vision of didactics of mathematics interested in characterizing the status of tables in society (their functionality historically, in information science, in statistics, in the curriculum, and in international evaluations), in identifying the actions that teachers carry out in implementing a data analysis learning situation dealing with making tables (the task proposed and the management of maintaining cognitive demands), and in discovering how learners gradually make tables to organize data in the first years of schooling (the progression of their representations).

We assume that school statistics is a new field of study, whose principal goal is to provide statistical literacy to students, education that demands of the teacher new perspectives and actions that allow learners to be able to interpret and critically evaluate information and develop the capacity to communicate their opinions regarding this information and thereby make informed decisions.

Specifically, as a result of schooling, it is hoped that students recognize and are able to interpret different representations of data, among them, tables. However, faced with the evidence of the absence of tables as a learning object, the question that guides this research is:

How do children learn tables?

The relations contained in tables allow for discerning the behavior of data associated with a context, and allow for responding to and communicating an issue. In the use of data tables in schools, difficulties are detected, but there is little research specifying the difficulties that subjects have in dealing with tables. The table links a physical structure, a relational positioning of data, and diverse use functions, with a statistical semantic content, all of which contributes to making learning and teaching tables more complex. So, we ask:

How does the notion of a table emerge in students in the first years of schooling?
How do students construct meaning from data?
What representations do students produce when faced with a new data analysis task?
What is the thinking behind the representations that students produce?
What levels of conceptualization are reflected in these representations?
What characteristics does a teaching task aimed at data analysis have?

How does the teacher manage a data analysis lesson in primary school?
How does the teacher maintain the task's level of cognitive demand?
What are the cognitive demands that tasks associated with tables create?
What are the components of a hierarchy of understanding tables?
Are the levels of understanding graphs the same as those for tables?

8. GOALS OF THE DISSERTATION

The general goal of this study is to reveal the table as a learning object in the first years of schooling in Chile.

As mentioned, trying to get citizens to acquire statistical literacy is a socially and didactically active issue. This research addresses an aspect of such a proposal, which is the frequency table used in school.

The table as a representation is diverse in contents, forms, and applications. Tables are widely used in various disciplines, especially as calculation and analysis tools in statistics and probability. The framework of statistical literacy in which we position this research is, at the same time, part of a larger framework of general literacy. In statistics, three types of tables are generally used: frequency tables, distribution tables, and contingency tables. In primary school statistics, the frequency table is used in its modality as a counting table, an absolute frequency table, a frequency table with other statistical calculations, and even a double-entry table that gives the frequencies of two variables. This work principally addresses one of the statistical representations used in schools: the frequency table in its most elemental definition.

Considering a didactic perspective -as a discipline related to the study of the processes of knowledge transmission- we want to study the historic evolution and epistemology of tables, the conceptualization of tables by subjects, and their processes of data analysis in working with tables, as well as explore some teaching proposals regarding frequency tables.

The principal goals of this study are:
1) Carry out an epistemological historic analysis of tables, identifying their diverse purposes in different times and cultures;

2) Characterize the cognitive process in data analysis using frequency tables;
3) Identify in teachers’ classroom management the maintenance (or lack thereof) of the cognitive demands that a frequency table task creates;
4) Configure levels of understanding of tables that help to explain the understanding of subjects faced with tables.

The didactic and cognitive components of this work enter in the framework of Garfield and Ben-Zvi’s *Statistical Reasoning Learning Environment*, in Vergnaud's *Theory of Conceptual Fields*, in Stein and Smith's *Levels of Cognitive Demand*, and in Wild and Pfannkuch's concept of *Transnumeration*.

For the first goal, regarding the epistemological component, we will enter in depth in the emergence of the table object and its role in knowledge development through an epistemological historical study. For the second goal, the representational functioning of tables will be analyzed (Vergnaud, 1990, 1994, 1996, 2007, 2013; Wild and Pfannkuch, 1999) to reveal subjects’ tendencies, difficulties, and patterns in the process of understanding. For the third goal, Garfield and Ben-Zvi’s Statistical Reasoning Learning Environment (Garfield and Ben-Zvi 2007, 2009; Ben-Zvi 2011) will be considered as well as Stein and Smith’s cognitive demands (Stein and Smith 1998, 2000), considering teachers’ learning communities in a lesson study. For the fourth goal, related to configuring a hierarchy of levels of table reading, categories will be defined based on the epistemological historical study and the analysis of table items in the TIMSS test (2003, 2007, and 2011).

Specific goals

With the theoretical framework outlined, we propose:

At the epistemological level related to tables:

Demonstrate their role as a tool from *proto-statistics* to modern days.
Define their role as a significant element in the analysis of knowledge circulation, as a storage repository and at the same time as a support for normalizing knowledge in antiquity.
3) Show evidence of their presence in diverse cultures or societies (Egyptian, Babylonian, Greek, Maya, and Inca) and their role as storage for administrative archives, as archives of

numbering and meteorology systems (in schools), and as scientific and mathematical archives (in academies).
4) Specify their role in the emergence and development of the concept of a function (their presentation in multidimensional arrays and interpolation techniques as representations of continuous phenomena).
5) Clarify their uses and roles in statistical activity, related to methodologically presenting a set of data or research results, as instruments for facilitating calculations, or as heuristic tools for exploring new situations.

At the cognitive level related to tables:

6) Identify tables as representations that support the construction of meaning for data.
7) Identify processes in the development of reading, interpretation, completion, and construction of frequency tables at the school level.
8) Define the ability to transnumerate data to obtain greater understanding of the data, by transforming raw data to a tabular representation.
9) Configure a hierarchy of levels (taxonomy) of reading specific to tables.
10) Determine whether the proposed taxonomy of understanding of tables behaves similarly to a taxonomy of understanding of graphs.

At the didactic (*sensu stricto*) level related to tables:

11) Study the role of statistical tables in the primary school mathematical education programs of study in three OECD countries.
12) Study the role of statistical tables in the primary school mathematical education program of study (2012) in the data and probability theme in Chile.
13) Study the role of tables in international tests according to the proposed activity: reading, interpretation, completion, and construction.
14) Elaborate, implement, and analyze a lesson that contributes to statistical reasoning through data analysis and the use of tables.
15) Elaborate, implement, and analyze a lesson centered on data analysis and statistical reasoning with high-level cognitive demands.

CHAPTER II

Literature Review and Theoretical Framework for the Research

"Le passage d'une démarche de pointage à une démarche d'interprétation globale dans la « lecture » des tableaux représente un saut du point de vue cognitif."
(Duval, 2003)

Chapter Summary

1. Introduction

This chapter first reviews the specialized literature relevant to tables in statistics. The focuses are the learning difficulties that the table format produces and the implications for teaching tables.

Next, the theoretical framework of the research will be given both for the cognitive and the didactic approaches. For the cognitive approach, Vergnaud's Theory of Conceptual Fields, Stein and Smith's Mathematical Task Analysis, and Wild and Pfannkuch's Transnumeration Process will be used. Additionally, fundamental statistical ideas will be detailed. For the didactic approach, a model of statistical education, Garfield and Ben-Zvi's Statistical Reasoning Learning Environment, will be presented through a Japanese lesson study.

The chapter ends with a reflexive and theoretical discussion.

2. Difficulties with the Table Format

Today's citizens live in an environment full of information. As such, statistical literacy is necessary as it provides people with the cognitive tools for filtering, understanding, evaluating and being critical consumers of said information. Acquisition of familiarity with basic terms and ideas related to graph and table presentations is a component of statistical literacy (Burrill and Biehler, 2011:60). Tables and graphs are part of the daily life of the citizens. However, according to the literature, despite the central role of tables in scientific practices and their widespread use in science classes and texts in diverse fields, interpretation of tables is a difficult task, and the acquisition of table interpretation abilities is not transparent. Conti and Carvalho (2011) point out that table interpretation abilities are not necessarily acquired through exposure to tables and should be taught since they are essential in today's society.

Previous research has shown the specificity of tables and students' and teachers' difficulties in dealing with them (Dibble, 1997; Brizuela, Lara-Roth, 2002; Duval, 2003; Martinez and Brizuela, 2006; Wu and Krajcik, 2006; Estrella, 2010; Gabucio, Marti, Enfedaque, Gilabert, Konstantinidou, 2010; Marti, Garcia-Mila, Gabucio, Konstantinidou, 2010a; Marti, Perez and de La Cerda, 2010b; Estrella and Mena, 2012). Some research shows that both the arrangement of sets of data in table form and the interpretation of tables are problematic for school children. This lack of competence in the use of tables can be explained by the lack of importance schools place on learning them, despite the fact that the curriculum includes them as teaching contents in mathematics and science in both elementary school and high school. However, the type of problems presented and the lack of a specific didactic sequence lead to tables being taught at a very basic level and, therefore, to students not recognizing tables as cognitive instruments (Brizuela and Lara-Roth, 2002; Martinez and Brizuela, 2006; Marti et al., 2010b; Gabucio et al., 2010).

Recent studies show that the organization of data in a double entry table causes difficulties for elementary school students and high school students. Marti et al. (2010a) report the level of competence that is required in the construction of a double entry table from a set of data and the main difficulties for students in completing the task. They identify and analyze the difficulties in terms of cognitive and graphic processes. Their results show that a relatively low percentage of students were successful in constructing a traditional table.

Didactic research about contingency tables has been specially focused on analyzing the capacities of different types of students to identify the connection or statistical relationship between two variables based on data presented in this type of table, both in the logical completion method and in reading them. Estrada and Diaz (2006) show that one of the incorrect strategies observed in reading contingency tables is to compare the cells two-by-two, and once it is admitted that absence-absence cases are also favorable for the existence of relationships, to use only data from one row or column.

In a research report about linear functions in first-year university students, Peralta (2002) states, “Despite the apparent success by the students, the table register on which this is based is disconcerting. The causes of this confusion appear to be related to the use of tabulation only as an intermediate tool that permits the localization of points in a chart based on an algebraic representation and not as a representation in its own right.”

In Pfannkuch and Rubick’s (2002) research, they affirm that the understanding the students ideally should have when examining a task with tables includes noticing patterns in the data table and determining the construction of the graph. In their experiment they do not observe these actions in concert, but rather isolated, where the majority of students use the table as a tool for making the graph.

In Diaz (2005) and Estrada and Diaz (2006) we see the complexity of the double entry table through a semiotic analysis. This analysis specifies the implicit mathematical objects used when interpreting the table and the main semiotic conflicts in the task. In the conclusions of the second study mentioned, the authors argue that due to the frequent appearance of tables in the media and in teachers' professional material [in Spain] and their role in decision making, it is necessary for the training of future teachers to consider the development of capacities in correct table reading.

Espinel and Antequera (2009) observed that subjects between 15 and 16 years old are able to read a table and to draw data from it, while it is more difficult for them to construct a table when having to consider more than one variable, having to consider rows and columns, especially if there is some order or priority for the correct construction.

Lehrer and Schauble (2000) investigated the progress of first, second, fourth, and fifth grade students in inventing and convening data structures to "mathematize" their classification activities. The authors found a bias in the spontaneous organization of data in non-overlapping categories and the students' problems with categories that provide information for each cell in the same column of a table. Lehrer and Schauble (op. cit.) found that recognizing, developing, and implementing criteria for an effective classification process is difficult for the students.

Nisbet et al. (2003) found that the process of reorganizing numeric data in frequencies is not an intuitive process for first and third grade children. Coutanson (2010) showed that inserting statistical data in a table causes a mental reorganization beyond a simple oral enumeration. He recommends a global apprehension of the table and in particular the organization of its margins in order to move beyond simple reading and access the interpretation of data.

On the other hand, Pfannkuch and Rubick[5] (2002) showed that there is little research about students constructing and interpreting statistical data tables and that it is necessary to research issues about how students perceive tables. The authors, when analyzing the change from one representation to another related to tables, consider that in the transition from data to tables, the comprehension process should go from one that is rich in context to one that is more dependent on statistical knowledge. They also consider that to produce a table there should be an abstraction of quantitative and qualitative variables from the context. Furthermore, they observe that the tabulation of data is an ability that requires determining a way of presenting the data clearly and without ambiguity, which implies some loss of information and the use of

[5] "*Little research appears to have been conducted on students' construction and interpretation of statistical data tables. This raises questions about how students perceive tables*."(Pfannkuch and Rubick, 2002, p.6).

variable descriptors. In relation to tables, the authors conclude in their analysis that, "*This was a skill that was being developed by all the students although it appeared to be a more sophisticated skill than we had realized"* (Pfannkuch & Rubick, 2002, p.13).
Duval (2003) affirms that Piaget (1955) showed that the possibility of constructing or developing a certain type of table occurs relatively late in the development of children's intelligence and that this is tied to the appearance of formal operational structures, that is, the appearance of combinatorial approaches.

Today's society and schools are characterized by the massive use of representations. *Statistical literacy* – recently introduced in primary school – in particular, considers the understanding and reading of at least graphs and tables. According to Echeverria and Scheuer (2009), these and other external representations are learning tools in the sense that one symbolic representation increases, "[…] several high-order processes that humans participating in complex social and cultural environments regularly carry out: communication, memory externalization, awareness and consciousness, learning and transmission, knowledge explicitation, adopting and sharing models or perspectives, extension of a sense of self " (p. 1).

2.1 Cognitive demands related to reading a table

A semiotic behavior involves interaction between "*sign, object, interpretant*" (Peirce, 1931). In relation to this, Kaput (1991) affirms that there is no notational result without an interpretative act realized by an interpreter. Furthermore, he affirms that this act requires knowledge a priori from the interpreter. Vygotsky & Luria (1994) noted that the availability of different cultural tools can alter or improve the applied memory strategies, the use of external symbols being a memory aide that might eventually be internalized by the subject and, in that way, memorization might be influenced by these external symbols even if the subject might not be producing them.
Duval (2003) maintains that the variety of types of tables and their different readings demand different cognitive resources and that making explicit that not all tables are alike and not all are read and understood in the same way is relevant. He adds that the students' difficulties related to tables might threaten other learning, and, therefore, it is necessary to offer types of

tasks that help students to get a better understanding of different types of tables and the different readings that are needed.

Duval (op. cit.) proposes a classification of the different types of tables in relation to the cognitive demands generated in their construction. Regarding the understanding of tables, he shows the existence of two levels: one focused on the interior of the cells and another focused on the construction of the table margins.

Duval further emphasizes that tables are not independent representations like those representations that privilege visualization. This is because the tables necessarily articulate, explicitly or implicitly, the representations of other registers. This lack of independence obliges the reader to read tables as representations of another register, and it is this *conversion* that justifies the reading manner (vertical/horizontal, diagonal or both). Duval underlines that tables are different from textual registers since they eliminate the syntactic organization of phrases and even though their layout permits a two-dimensional localization, it is also different from maps and graphics since they do not permit readings similar to the readings of those.

A subject reading the data in a table has various ways of accessing the data, even though he or she unquestionably needs to look at a cell, which changes the reading from being basic to being more complex.

A subject in search of information contained in a table accesses the areas of the table, and visualizes it in terms of the physical and logical relations among its basic elements with the *functional* purpose of finding the information. After searching, the subject might reach interpretative levels beyond only pulling out information.

A subject that interprets a table in its totality might begin reading through the logical structure of the table, considering the restrictions of two-dimensionality, detecting the organization of the cells as an indicator of the relation between them, the intention of the author of the table, and the explicit and implicit references, among other things.

A reader who interprets a table is a semantic subject who articulates form and content, reorganizing the syntax of the phrases, and who searches for the meanings of the content through the localization of cells in the table.

Giving examples of the readings of a table provides us with some features of the complexity of the table and the processes of understanding that it requires. Furthermore, tables appear to show various forms and content according to the area where they emerge. This influences learning tables in different levels of school and how they are taught.

2.2 Consequences for how tables are taught and learned

In most countries, new curriculum changes have made the teaching and learning of statistics a crosscutting topic for all grade levels. In pre-school and basic education teachers' training, statistics and probability contents have slowly been integrated into the curriculum. However, existing teachers, in addition to having neither studied nor practiced teaching this type of content or its representations, do not have experience or access to information that lets them understand the difficulties in understanding the usual formats in the scientific area like, e.g., the table format, which requires different and complex processes, at least in reading, construction, completion, and interpretation (del Pino & Estrella, 2012).

Sanz (2001) investigates tables in textbooks and maintains that the underlying relation can be anything, from a similarity relation to order and equivalence relations, order being the most typical. The author applies semantic criteria to distinguish between two types of tables: data tables and operations tables. She informs that textbooks do not distinguish between charts, graphs, and tables, and that it is the context that determines whether something is a data table or an operations table, which means the significance of these tables is, at least partially, determined pragmatically.

On the other hand, there is a belief that a table is a simple structure that only involves organizing numbers in horizontal rows and vertical columns. Furthermore, there is a belief that untrained users easily understand that there is an implicit commonality among all the numbers in the same row, often belonging to a "case", and among all the numbers in the same column, often representing a "variable" (for example, Koschat, 2004). In teaching this type of representation, teachers should keep in mind that some tables correspond to the juxtaposition of lists, while represent the crossing of lists (Coutanson, 2010).

To help students develop the abilities mentioned and become statistically literate citizens, mathematics and science teachers should promote data analysis and activities related to registering and classifying data in the classroom. Statistics education proposes that the initial emphasis should be on exploratory data analysis and that only after the students are familiar with data analysis can they come to evaluate the importance of design (Cobb & Moore, 1997).

In this sense, to the student's question of, "How do I make a table?" the teacher's counter-question is "What do you want to show in the table?" At all educational levels, the educators should propose practices that foster the development of table and data analysis competencies by providing opportunities to reflect on findings, explain their thoughts, their interpretations, and their misunderstandings with their peers in shared discussion spaces, compare results and write down their explanations.

3. Cognitive Approach

Since in order to create teaching situations that foster table learning it is necessary to identify the main elements in the process of conceptualization of a table, the Theory of Conceptual Fields (TCF) was applied. This is a theory that Gerard Vergnaud has been developing since 1981.

This cognitivist theory was chosen, firstly, because of its potential for providing the researchers with a theoretical framework for complex cognitive activities, especially those related to scientific learning, as it allows us to analyze the relation between concepts such as explicit knowledge and implicit operative constants in the behavior of the subject in situation. Secondly, we have chosen the TCF since we consider it a pragmatic theory that looks for practical consequences of thinking, in the sense that knowledge acquisition is shaped by problematic situations and by learning actions in these circumstances.

For a table learning situation created in a statistics education model that promotes statistical reasoning, we have chosen to evaluate the teacher's mediation act by observing the cognitive demands that he or she promotes. In this research Stein and Smith's the theoretical framework of Mathematical Task Analysis is also used (1998; Stein, Smith, Henningsen, & Silver, 2000). This theory was developed by Stein and Smith in order to establish how the teacher develops statistical tasks and how they are carried out by students.

In addition to the above, and in order to address the characteristics of statistical reasoning and the processes that are activated when changing representations, we have chosen to focus on

the cognitive levels and fundamental statistical ideas, as mentioned by Garfield and Ben-Zvi (2008) and Wild and Pfannkuch's (1999) capacity for *transnumeration*.
In the following, we present the fundamentals of the approaches mentioned.

3.1. The learner's perspective: Vergnaud's Theory of Conceptual Fields

Piaget's works, which contributed to understanding concept formation in children and teenagers, including concepts of order and logical classes, does not consider learning in school. As Vergnaud (2013) points out, didactics research is what combines teaching and learning a discipline with mastery the knowledge involved. For the Vergnaud's Theory of Conceptual Fields (1990, 1994, 1996, 2007, 2013), it is crucial to distinguish between operational knowledge (what is done in situation) and predicative knowledge (what is said). The first hypothesis in specific didactic research assumes that the acquisition of the meaning or meanings of a concept occurs based on confronting problematic situations that put the concept into play; it is the situation that gives meaning to the concept.

Conceptual Fields developed as a means for understanding the role of experience, where the situation and *schemes* are central to addressing the study of complex competencies that are formed progressively and that involve a variety of concepts and symbolic representations. The TCF conceives conceptualization as found in action and considers conceptualization to be the identification of objects at different levels, noticeable or not, of their properties and relations in the course of the activity in situation. Symbolic and linguistic means of representation developed in the culture allow the identification of constructed concepts and thus the signifier/signified systems allow for expressing the properties of the objects and actions and communicating about them, providing conceptual constructions with stability and aiding implicit conceptualization in an action.

Vergnaud points out that these systems do not exhaust the categories of thought that function in an action, since much can be said about objects and actions, and this evidently contributes to their conceptualization, but it does not stop here. Operational forms of knowledge are always richer than predicative forms, even if the latter are more analytic. At the same time, verbal symbols and the symbols that are often used today in science and technology (diagrams, tables, graphs, algebra) play a crucial role in conceptualization: the invariance of the symbolic form comes to the rescue of the invariance of the concepts, and the symbol in particular allows us to talk about objects and properties that are not accessible to direct perception (Vergnaud, 2000:17).

Vergnaud considers a *scheme* as the invariant organization of behavior for a certain class of situations, which looks for knowledge in action so that its actions are operative. Schemes generate activity in function of the situation's variables and organize the observable activity and, more importantly, the thinking activity that lies behind it. Individual schemes are dynamic and functional, and they develop progressively, so they change over time.

Vergnaud defines the concept C(S; IO; SR), where the situation [S] is a set of situations that gives meaning to the concept, that challenge it or make use of it. The meaning [IO] is a set of operational invariants (concepts and theorems in action) upon which the scheme rests. The concepts in action are relevant categories and the theorems in action are propositions considered true. The concepts and theorems can be implicit or explicit, more or less formal, correct or incorrect. The significant [SR] is a set of linguistic and non-linguistic forms that allows us to symbolically represent the concept and its properties and processing procedures.

The identification of the theorems in action used by the children permits us to design learning situations that can bring the children's knowledge closer to the objective knowledge. Vergnaud (1990) affirms identifying the theorems in action is a way of discovering the intuitive knowledge, connecting them with explicit mathematic content and providing recommendations for teaching. The TCF explicitly recognizes the existence of mathematical knowledge that lies behind the activity. In situation, the children discover new aspects and eventually new *schemes*. Since the formation of operational invariants is the basis for conceptualization in the course of the activity, and these shape the scheme's operationality, in this case, it is necessary to study the conceptualization of tables in order to be able to choose from possible learning situations in order to make the concepts and theorems in action emerge, at least partially, and so that the invariants become explicit rational conceptions.

In the TCF, the teacher has a fundamental mediating role in the explicitation process of implicit knowledge, in which the theorems in action and concepts in action can become true scientific theorems and concepts. An explicit proposition can be debated, a proposition seen as true in an implicit way cannot.

3.2. The teacher's perspective: Stein and Smith's Mathematical Task Analysis

The importance of tasks in the development of a mathematical way of thinking has been recognized in various approaches. Some authors believe that the tasks used in the classroom determine the type of learning that students create (Stein & Smith, 1998; Stein, Smith, Henningsen & Silver, 2000). The use of tasks that execute routine memorization processes makes students develop a certain form of thinking that is different from that which they would develop if they faced tasks in which they must think and reason conceptually and establish connections. These authors maintain that a high-level cognitive task can be maintained at the same level by the students or be brought down to a lower level, while a low-level cognitive task cannot be converted into a high-level task by the students. The study allowed them to elucidate the factors associated with the maintenance and decrease of high-level cognitive demands (see Tables 1 and 2).

In the framework of mathematical tasks (Stein & Smith, 1998) an important characteristic of these is established: the level of cognitive demand. Stein, Smith, Henningsen and Silver (2000) define the cognitive demand of a mathematical task as the type and level of thinking required of the students in order to successfully participate and solve the task. The cognitive demand serves as a way of classifying the intellectual machinery that a task stimulates in the students.

In tasks with a low level of cognitive demand, memorization and the use of simple methods or algorithms without relation to other ideas or mathematical concepts are favored, while in tasks with a high level of cognitive demand, carrying out processes that activate conceptual reasoning and the construction of relations between the concepts involved in resolving a problem is favored. Activities with a high level of cognitive demand allow the student to *do mathematics* – in this case statistics – which means that the student develops activities that involve examining, observing, conjecturing, detecting relations and constants, as well as explaining and communicating results (see Table 1).

Table 1
Task Analysis Guide (Stein & Smith, 1998; Stein, Smith, Henningsen & Silver, 2000)

Cognitive Level	Type of task	Characteristics
Low-level demands	Memorization tasks	Reproduce what was learned previously concerning facts, rules, formulas or definitions. Are solved without using procedures or the time is not sufficient to use procedures. Reproduce exactly the material seen before, which is clearly and directly stated. There is no connection with the concepts or the significations that lie behind the facts, rules, formulas or definitions that are being learned or reproduced.
	Procedures without relation to the tasks.	Become algorithms since the use of procedures is specifically requested or their use is based on prior instructions. Require limited cognitive demand for completing successfully. Are unrelated to the concept or to the meaning that underlies 0the procedure being used. Are focused on the production of correct answers instead of developing mathematical (statistical) understanding. Do not require explanations or the explanations focus only on the description of the process used.
High-level demands	Procedures with relation to the tasks	Students' attention is focused on understanding mathematical (statistical) concepts and ideas. Paths are suggested (explicitly or implicitly) that are general and broad procedures closely related to the underlying conceptual idea. Establish relations between multiple representations that help to develop the meaning in multiple ways (e.g. visual diagrams, manipulatives, symbols, problem situations.)
	Doing mathematical (statistical) tasks	Require some form of cognitive effort to understand the conceptual ideas that underlie the procedures. Require complex thinking, not algorithmic thinking. Require that the students examine and understand the nature of the mathematical (statistical) concepts, process or relations. Demand self-monitoring or self-regulation of one's own cognitive processes. Require students' relevant knowledge and experience to be used correctly to resolve the task. Require students to examine the restrictions of the task that might limit the strategic possibilities for solving it and the solutions. Require a considerable cognitive effort and might involve a certain level of anxiety from the student.

Table 2
Factors associated with the maintenance or decline of high-level cognitive demands (Stein & Smith, 1998; Stein, Smith, Henningsen & Silver, 2000)

Factors of maintenance or decline	Explanations
Factors associated with the maintenance of high-level cognitive demands	
M1	Scaffolding for students' thinking and reasoning is provided.
M2	Students are given the means to monitor their own progress.
M3	Teacher or capable students model a high-level performance.
M4	Teacher applies pressure so that justifications, explanations and meanings emerge through questions, comments and feedback.
M5	Tasks are based on students' previous knowledge.
M6	Teacher realizes frequent conceptual connections.
M7	Sufficient time is given for exploring, neither too little nor too much.
Factors associated with the decline of high-level cognitive demands	
D1	Problematic aspects of the task become routine.
D2	Teacher adjusts the emphasis from meaning, concepts or understanding to the correctness or completeness of the answer.
D3	Sufficient time is not given to deal with the most demanding aspects of the task.
D4	Classroom management problems prevent sustained commitment in high cognitive level activities.
D5	Task is inappropriate for a given group of students.
D6	The students do not need to deliver product accounts or high-level processes.

We use Stein and Smith's (1998; Stein, Smith, Henningsen, & Silver, 2000) Mathematical Task Analysis in order to identify how the teacher sets up the statistical task with regard to the level of cognitive demands that he or she promotes in the students.

3.3. Statistical Reasoning and Thinking

In the last decade, three important concepts have appeared in the field of statistics education: *statistical literacy, statistical reasoning,* and *statistical thinking*. This cognitive hierarchy was initially developed by Garfield (2002) and originated in learning statistics at the undergraduate level (higher education). In the following, some of the characteristics of each cognitive level are identified.

Statistical Literacy

Statistical literacy involves the understanding and use of basic statistical language and tools: knowing what statistical terms mean, understanding the use of statistical symbols, and recognizing and being able to interpret data representations (Rumsey, 2002). Ben-Zvi and Garfield (2004) make *statistical literacy* more specific by including important basic abilities that are used to understand statistical information, such as the capacity for organizing data, constructing and presenting tables, and working with different data representations. They also include understanding of concepts, vocabulary, and symbols and understanding of probability as a measurement uncertainty.

Statistical Reasoning

Ben-Zvi and Garfield (2004) establish that this type of reasoning can be defined as what people do when reasoning with statistical ideas and giving meaning to statistical information. This interpretation implies making decisions based on data sets, data representations, or summary measures of the data. Statistical reasoning can relate one concept to another (e.g. center and dispersion) or can combine ideas about data and chance. This reasoning also means understanding and being able to explain and interpret statistical processes and results completely.

Statistical Thinking

If statistical reasoning is understood as the way in which a person reasons with ideas and gives meaning to statistical information, then statistical thinking involves higher order thinking abilities even greater than statistical reasoning. It is the statistics professionals' way of thinking and includes knowing how and why to use a particular method, measure, plan, or model statistically.

Statistical thinking involves understanding why and how statistics research is developed. This includes recognizing and understanding the research process in its totality (from the question raised, collecting data, analysis choice, test assumptions, etc.), understanding how models are used to simulate random phenomena, understanding how data are used to estimate probabilities, and recognizing how, when and why the existing inference tools can be used, and involves being able to understand and use a problem's to plan and evaluate research and draw conclusions (Chance, 2002).

Figure 1 (Estrella, 2010) is developed from Garfield and Ben-Zvi's article (2007). In the figure, the characteristics that distinguish statistical thinking are presented.

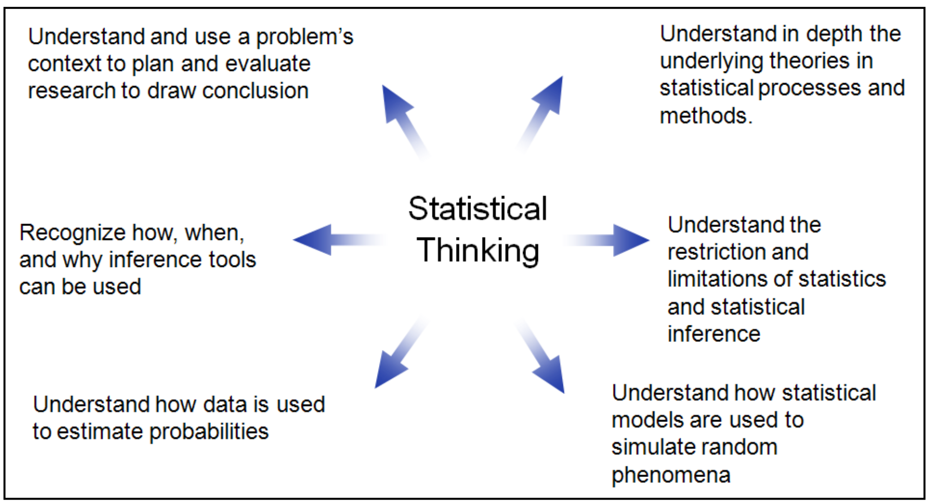


Figure 1. Characteristics of Statistical Thinking

The foundations of statistical thinking, as described by Wild and Phannkunch (1999), emerge in the action of the five following process:

(1) recognizing the need for data;

(2) *transnumeration*;

(3) perception of variation;

(4) reasoning with models, and

(5) integration of statistics and context.

The fundamental component, *recognizing the need for data*, is the basis for statistics research. It is the hypothesis that many situations in real life can only be understood based on an analysis of data that has been collected the right way. Personal experience and/or anecdotal evidence are not reliable and can lead to confusion in judgments or decision making.

The second component is *transnumeration*. This is a term that indicates the understanding that arises in the dynamic process of changing representations in different registers.

The third component is *perception of variation.* This component requires understanding the variation that exists and that is transferred to the data as well as understanding the uncertainty that comes from unexplained variation. Variation affects making judgments based on data, since without understanding that the data varies based on possible patterns and tendencies, people tends to express generalizations based on a particular group of data as certainties, not probabilities.

The forth component is *reasoning with statisticak models*. This component is related to any statistical tool that represents reality, but differentiates the data from the model and at the same time relates the model to the data. In order for people to be able to give meaning to the data, statistical thinking requires the use of models. At the school level, the appropriate models with which students could reason include graphs, tables, and summary measures (such as median, average, and rank).

The last fundamental component of Statistical Thinking is the *integration of statistics and context*. This component highlights making connections between contextual knowledge and the results of the statistical analysis to reach meaning. Constantly linking contextual knowledge of a situation being researched with statistical knowledge related to the data in the situation permits us to give meaning to the data and to understand it better. Therefore, this is a sign of high-level statistical thinking. This integration helps students to understand that statistics is not developed in isolation from the real problems; it deals with numbers in context.

3.3.1 *Transnumeration*

This term, coined by Wild and Pfannnkuch in 1999, refers to the process of “changing representation to generate understanding” (1999, p. 227). *Transnumeration* responds to the fact that sometimes in the data analysis and organization phase a representation can unveil something new that was hidden before and in doing so provide a better understanding of the problem.

Pfannkuch and Rubick (2002) maintain that reasoning with data is a complex task that requires students' imagination to produce a network of connections between the contextual knowledge and the statistical knowledge. This integration allows students to construct meaning based on data through a constant dialog in a range of statistical representations. The authors describe *transnumeration* as thinking about how to change the current representation to another representation; for example, it could involve thinking about reclassifying data or representing the data in tables or graphs.

In our research, we assume that representations such as tables, graphs, words or symbols produce conceptual relations individually and in relation to other representations. The understanding that is produced by transforming written language into a table differs from the understanding that comes from transforming a table into written language. To differentiate transnumerical understanding processes, with regard to a change of representation, Table 3 summarizes the processes activated in changes between representations, highlighting the processes where tables are involved.

Table 3
Cognitive processes between representations (adapted from Janvier, 1987)

To \ From	**Written language**	**Table**	**Graph**	**Algebraic language**
Written language	Diversity of descriptions of the relations	Organization, classification and spatial arrangement of the relations	Outline of descriptions of the relations	Modeling of the relations
Table	Reading and interpreting relations between variables and value categories	Modification of relations or creation of new tables	Value relations of variables with localizations in a cartesian plane or area of the pie chart	Searching, writing, and adjusting the relations
Graph	Interpretation of tendencies and graphic relations	Interpretation of relations and extractions of values and categories to tabulate	Modifications of relations or creation of new graphs	Searching, writing, and adjusting the relations displayed
Algebraic language	Reading and interpretation of symbolic relations	Generation of relations between values and variable categories (calculations)	Representation of relations in graphic forms	Adjustments in relations

Pfannkuch and Rubick (2002:5), based on Wild and Pfannkuch´s (1999) three aspects of transnumeration, point out instances of transnumeration in statistical thinking: taking measurements that capture the qualities or characteristics of the real situation; transforming the raw data into other representations, such as organized data, graphs, tables, and summary statistical measurements, to search for the meaning of the data; and communicating this meaning to others in terms of the real situation.

Rubick (2004) establishes four phases of transnumeration: recognition of the data's message; election of representation; transformation of the data and representation of the transformed data. Based on his research, he shows that even though many students were able to produce representations effectively, others had difficulties with aspects of the transnumeration process, many at an arithmetic calculation level.

Chick (2004) investigates the abilities necessary for representing data and proposes a framework of transmumerical techniques, specifying two of the three aspects of transnumeration concerning representations and communication. The author found that even though students had techniques for representing data, the election of the type of graph was not always adequate and that they overlooked simple techniques such as sorting and grouping data, something that could have helped the clarity of their representations.

The transnumeration framework proposed by Chick (2004) applies to data with the objective of discovering and showing its behavior (see Table 4). Each transnumerical technique involves some "change in the representation", creating a new variable, organizing the data in a different way, or representing it visually. Graphing, tabulating, and changing representation are all types of transnumeration.

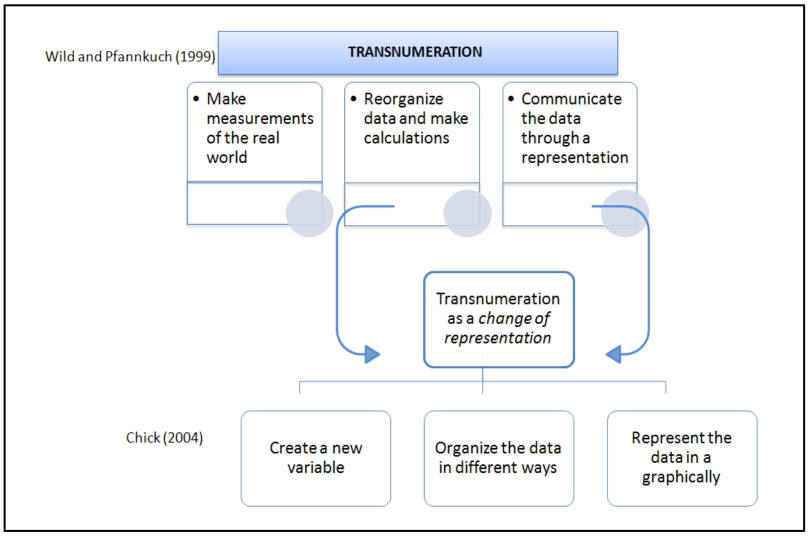


Figure 2. Wild and Pfannkuch's (1999) transnumeration and Chick's (2004) proposal.

In general, these representations are produced in the final stages to show the results of data analysis, but they are also created during prior exploration of the data. Chick emphasizes that other transnumerical stages precede graphing and tabulating, such as transforming the data to represent it (calculating frequencies in the data, realizing counts).

Table 4
Framework of transnumerical techniques (Chick, 2004)

Technique	*Description*
Organizing	The data is organized by some criteria. New variables do not arise.
Grouping	The data is grouped according to some criteria. This creates new variables. The change of variables might involve some type of transnumeration in advance.
Election of subset	A subset of data is chosen for a transnumeration.
Change of variable type	A numerical variable is thought of in categorical terms or a categorical variable is thought of in numerical or ordinal terms.
Calculation of frequencies	Frequencies of occurrence of the values of a categorical variable. Creates a new variable.
Calculation of proportion	Proportion, fractions of a whole. This creates a new variable.
Graphing/tabulating	Some or all the variables in the data (in its current form) are graphed or tabulated.
Calculation of central tendency	Measurement of a variable's central tendency (average, e.g.). May create a new variable.
Calculation of dispersion measure	A dispersion measurement of the values associated with a numeric variable. May create a new variable.
Other calculations	Generic term, recognizes that other statistical calculations of the data are possible (sum, correlation coefficient, e.g.)

3.4. Fundamental Statistical Ideas

In recent years, the development of statistics education has led to specifying some of the key statistical ideas, including: data, distribution, variation, representation, association and modeling of relations between two variables, probability models, and sampling and inference (Garfield & Ben-Zvi, 2008; Burrill & Biehler, 2011).

The data is the heart of statistical work, the subjects need to understand that data are numbers within a context, recognize the need for data in order to make decisions and evaluate information, recognize the different types of data, and understand how the methods to gather them (via surveys) and produce them (in experiments) make a difference in the types of conclusions that can be found; furthermore, knowing the characteristics of good quality data helps to prevent bias and measurement errors.

The concept of *distribution* lets us understand that a set of data can be examined and explored as an entity (a distribution) more than as a set of separate cases, that a graph of this data (quantitative) can be summarized in terms of form, center and dispersion, that different representations of the same data might reveal different aspects of the distribution, and that distributions can be formed from values of individual data or from summary statistics such as an average (that is, distributions of sample averages).

Variability is an omnipresent characteristic in the cycle of statistics research, given that it allows us to understand that the data varies, sometimes in a predictable way. There are sources of variability that can be recognized and used to explain it. Variability is sometimes caused by random sampling or measurement errors, other times, by the properties of what is measured. An important part of data analysis is determining how divergent the data in the distribution is. Usually it helps to know a center measurement when interpreting variability measurements. The election of these measurements depends on the form and other characteristics of the distribution. Different variability measurements show aspects of the distribution (e.g., standard deviation focuses on the typical distance from the average, the range gives the difference between the minimum and maximum value, and the IQR gives the width of the central half of the data). The idea of the *center* of a distribution as a “signal in a random process” considers the comparison of two sets of data to make decisions (Konold & Pollatsek, 2002). This is a “signal” that can be summarized by a statistic measurement, such as average or median. It is very useful to interpret a center measurement together with a dispersion measurement, and these decisions are often based on the form of the distribution and whether or not there are other characteristics such as outliers, groupings, gaps or asymmetries.

Statistical *models* are useful when explaining or predicting the value of the data. Comparing the data with a model to see how well the data fits, examining the model's remainders or deviations. It is also possible to use models to simulate data in order to explore properties of procedures or concepts.

Randomness permits us to understand that each result of a random event is unpredictable, but that we can still predict patterns in the long run.

The concept of *covariance* lets us understand that the relation between two quantitative variables might vary in a predictable way. Sometimes, this relation can be modeled, which lets us predict the values of a variable using the values of another variable.

Sampling indicates that a lot of statistical work includes taking samples and use them to estimate or to make decisions about the population they come from. The samples extracted from a population vary. Freundenthal (1974, cited by Burril and Biehler, 2011) points out that the importance of sampling in statistics is the variation between samples and that this variation is reduced when the sample size increases. Statistic *inference* permits us to make estimations or decisions based on the data samples in observational and experimental studies. The accuracy of the inferences is based on the variability of the data, the sample size, and the appropriateness of the assumptions that underlie random data samples, e.g., independence or equiprobability.

Regarding the idea of *representations*, in this research tables are conceived as a representation of data in statistics, and as a relational structure that allows us to organize and visualize data, which can provide knowledge about the data's behavior in relation to distribution and variability.

After presenting the theoretical framework of the research for the cognitive perspective, in the following we present the framework for the didactic perspective.

4. Didactic Approach

In order to facilitate students' conceptualization processes and to produce a table learning situation in statistics education, the educational model known as Garfield and Ben-Zvi's Statistical Reasoning Learning Environment (SRLE) will be adopted, both for the student and the teacher. Furthermore, the Japanese Lesson Study will be taken into account. This will guide the construction of a lesson plan.

4.1. A statistics education model

Abundant research in statistics education in recent decades and articles by leading statistics professionals have helped to create a paradigm change in the conceptualization of statistics education. Pfannkuch and Ben-Zvi (2011) indicate that a reform in teaching has evolved based on technological advances, on the identification and specification of the characteristics of statistical thinking, and on the "big ideas" that sustain statistics. The researchers point out that the explanation and exploration of these ideas have contributed to approaches that emphasize exploratory data analysis (EDA), paying attention to building conceptual understanding in students, and curriculum with the goal of developing students' reasoning, thinking, and literacy.

Students and teachers need to understand several of the fundamental statistical ideas at a deep conceptual level. These ideas serve as a guide in teaching and in students' learning. Among other concepts, these ideas include data, variability, distribution, center, representation, statistical models, randomness, covariance, sampling, and inference.

The students' statistical reasoning comes from statistical knowledge, problem context, social norms, progressively developed habits, and the support of an inquiry-based environment that includes activities, tools, and scaffolds (Makar, Bakker & Ben-Zvi, 2011). In particular, it comes from activities that provide the students with opportunities to think and reflect on their learning and debate and reflect with their classmates. This focus presupposes that learning is improved when the teacher pays attention to students' prior knowledge and beliefs and uses this knowledge as a starting point for new teaching, monitoring the changes in conceptions as teaching advances.

Ben-Zvi (2011) considers that generally it is easier to prepare a lecture than to design a learning environment where students participate in activities and discussions and/or collaborative projects with help from technical tools. The author affirms that while the first approach is focused on the teacher –"What do I want to say to my students?"; "What do I want to cover?"; etc.– the second approach is more focused on the student: "What can I do promote students' learning?"; "How can I involve students in the learning, in practical activities, in developing reasoning, in discussions of ideas, and ingroup work?"; etc. Using the second approach, the teacher presents him or herself as a learning facilitator instead of leading the acquisition of knowledge in a teacher-centered lesson.

The main reason for changing to the student-centered approach is that it is more effective in helping students to build a deeper understanding of statistics and in transferring what is learned to future classes or to real life. A problem with the teacher-centered approach is that students rarely get the opportunity to develop a deeper understanding than what they have "learned", which they quickly forget (Ben-Zvi, op.cit.).

4.1.1 Garfield and Ben-Zvi's Statistical Reasoning Learning Environment

The Statistical Reasoning Learning Environment (SRLE) is an environment that promotes reasoning statistically and that seeks to develop a deep and significant understanding of statistics in students (Garfield & Ben-Zvi, 2008).

This environment combines materials, activities, class norms, scaffolding, discussion, technology, learning focus, and evaluation. The model is based on 6 teaching design principles (Cobb & McClain, 2004, cited in Ben-Zvi, 2011) (see Figure 3):

1. Focus on developing understanding of fundamental statistical ideas.
2. Use real data and motivators that involve students in elaborating and testing statistical conjectures and inferences.
3. Use collaborative class activities based on inquiry to help develop students' reasoning.
4. Integrate the use of technological tools that permit students to test their conjectures and examine and analyze data interactively.
5. Promote class norms that include statistical discourse and argumentation based on fundamental statistical ideas.
6. Use alternative evaluation methods to understand what students know and how they develop statistical learning.

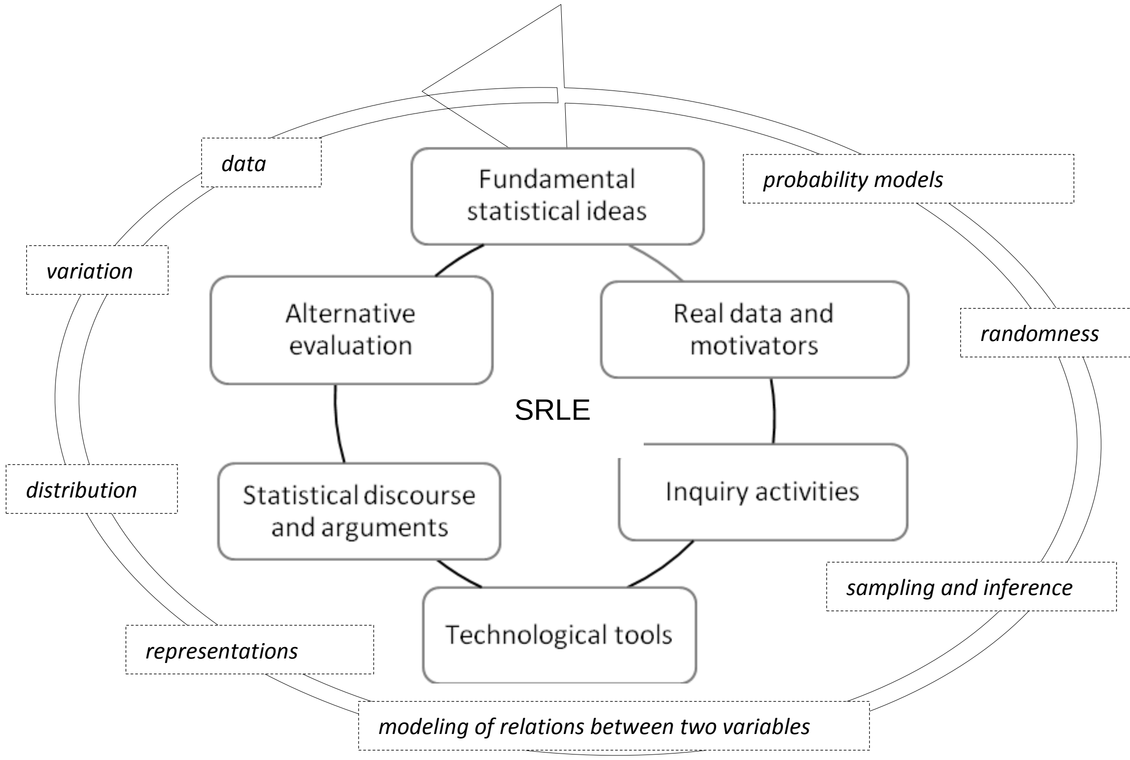


Figure 3. Six key elements of the Statistical Reasoning Learning Environment (SRLE) and fundamental statistical ideas.

Pfannkuch and Ben-Zvi (2011) propose extending the SRLE model to a teacher training course with the following components: deepening the understanding of key statistical concepts, developing the ability to examine and learn from data, developing statistical argumentation, using formative evaluation, and understanding students' reasoning (see Figure 4).

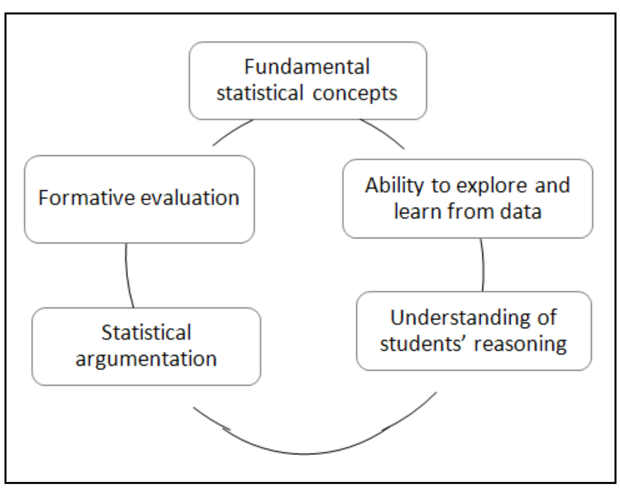


Figure 4. Five fundamental components for teacher education (Pfannkuch and Ben-Zvi, 2011)

4.2. Lesson Study

Unlike in western cultures, in both Japan and China the mathematics teacher's classroom teaching is open for observation by colleagues and for study and discussion. In Japan, lesson studies have been done for 140 years, both with the goal of improving scholastic learning in classes and the purpose of favoring teacher and curriculum development. Today, different formats exist: some lesson studies are carried out by teachers in their own educational community and are observed by their peers; others are carried out by expert teachers in big auditoria, sometimes receiving hundreds of teachers observing a lesson that displays innovation and excellence. Whatever lesson study model is used, there is always an implementation of a lesson open to observers. When the lesson is over (*research lesson* [or implemented lesson]), a debate is opened through which critical points regarding the discipline and the proposal are analyzed and a possible future implementation is planned, generally with another teacher in charge and with other students (Figure 5).

In a lesson study, a group of teachers initially *prepare* a lesson or a set of lessons, choosing relevant materials to reach the defined goal. Later, they discuss and design their lesson plan. Afterwards, one of the teachers that participated in the design *implements* the lesson study or research lesson, in which the design group participates as critical observers. Furthermore, other professionals from the educational field might witness the class. In the implementation of the lesson, the teacher starts with a review of the previous lesson, then he or she presents the day's problem, a challenge in the form of a question. The students must understand the problem, and then work individually or in groups, discussing possible methods of solving the problem, and finally exchange experience through discussion and argumentation, which leads to conclusions.

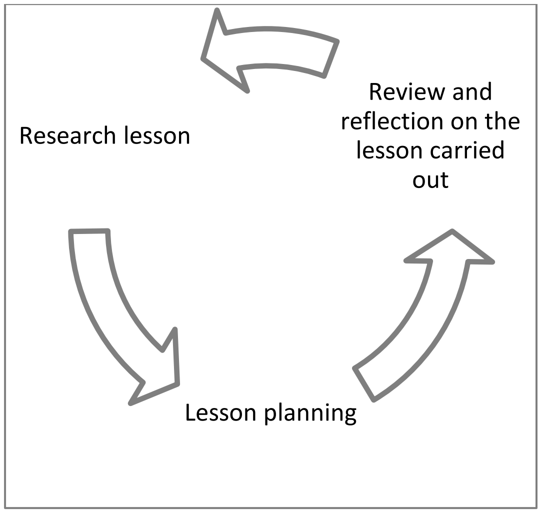


Figure 5. Phases of a lesson study

After completing the class, the teachers, and generally the observers, meet for a critique session to *review and analyze the lesson carried out*, which is projected into a future implementation that allows them to constantly improve the design and the implementation of the session.

In the lesson study and in the lesson plan, it is possible to highlight some repetitive processes that the teacher should realize. Those are: monitoring, anticipating, and selecting. This means monitoring how students examine and solve the task and the strategies and representations they use; anticipating errors and task demands, which requires the teacher to have done the task looking for more than one possible solution (both correct and incorrect); and choosing which students will share their strategies and representatoins, based on the monitoring of the lesson.

The model's components are demonstrated in the operationalization of the lesson plan that the model demands. In our case: the problematic statistical situation and the key questions (fundamental statistical concepts, real data, and inquiry), teacher intervention (monitoring, examining, and learning from data and statistical argumentation), and the evaluation of the lesson in action (formative evaluation).

5. Chapter Summary

In this chapter, we have reviewed tables' habitat. We started by presenting some findings regarding difficulties in dealing with tables in the statistics education literature. We characterized some of the roles of subjects faced with table tasks in order to specify the underlying cognitive complexity, and we outlined some of the consequences for learning and teaching tables.

As didactics investigators, our research position addresses a system: school knowledge – tables in data analysis –, students' role in conceptualizing tables, and the teacher's role in activating high-level cognitive demands in class to promote fruitful learning by students. We aim to study, within the variety of possible types of tables at school level – the frequency table –, the cognitive functions these fulfill and the educational problems raised by their use.

We have assumed theoretical frameworks in this research (see Figure 6); to characterize the cognitive processes in texploratory data analysis when primary-school students use frequency tables, the Theory of Conceptual Fields was adopted as a cognitive theoretical framework. This theory lets us identify the process of conceptualization of frequency tables that students build in situation.

Furthermore, considering that for any learning situation it is didactically interesting to evaluate the teacher's mediating actions, and even more so for issue in the Chilean context, we have used the Mathematical Task Analysis theoretical framework in order to establish the level of cognitive demands that the teacher manages when establishing a statistical task and guiding the development of the lesson.

Our interest lies in statistics education, and in order to address the characteristics of statistical thinking and the processes that are activated when changing representations we have adopted the cognitive levels and fundamental statistical ideas described by Garfield and Ben-Zvi (2008) and Wild and Pfannkuch's (1999) *transnumerative* ability, respectively. Considering that we research a scholastic frequency table situation, we have adopted the SRLE learning model and the methodology of Japanese Lesson Studies, carried out by the teachers themselves in order to improve the lesson.

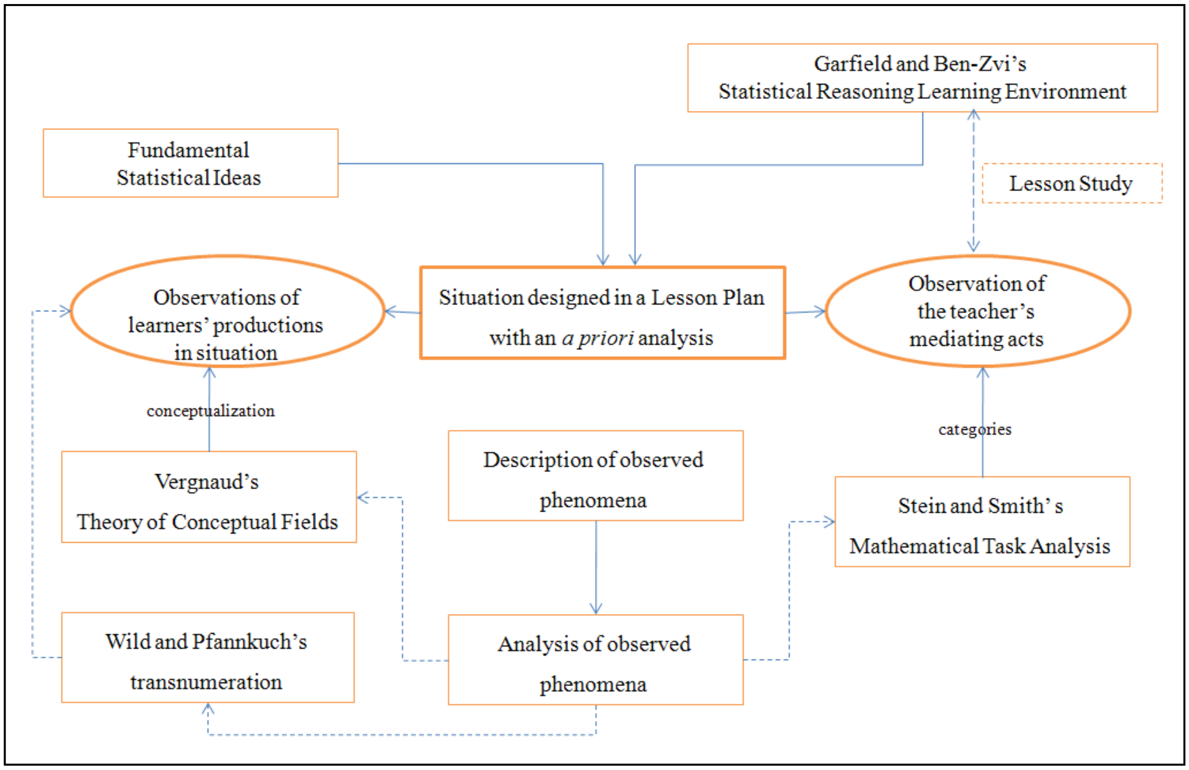


Figure 6. Theoretical frameworks assumed in the didactic system under investigation.

CHAPTER III

TOWARDS AN EPISTEMOLOGY OF TABLES

« L'historien des sciences doit prendre les idées comme des faits.
L'épistémologue doit prendre les faits comme des idées,
en les insérant dans un système de pensées. »

(Bachelard, 1977).

Chapter Summary

This chapter provides an overview of the process of historical evolution of ideas on the table, its connotation of human tool, and its emergence and development in different cultures and different times in history, issues that contribute to the knowledge on this subject and its teaching scope.

This part of the thesis aims to shape a vision of the role of tables in the construction of certain milestones of mathematics, and outline their epistemological path up to their current status of mathematical objects.

Specifically, the chapter addresses the trajectory of the table and its presence in different cultures as a storage tool, as a calculation tool in numbering systems and metrology, as a tool of analysis in scientific and/or mathematical fields, and its relation to the genesis and use of the concept of function.

1. INTRODUCTION

The study allowed us to observe that, as the concept of number, the tables appear in different times, places and cultures. The process of appearance is large and complex, and it shares with the generality of the development of other fields of thought a failure to be cumulative and continuous. However, its origin in the practices of societies is so evident that even the necessarily linear review we present here to highlight some aspects of interest for its teaching can show both their progress throughout history (and further back) and the social needs to which they respond.

We have gathered some relevant background, especially of numerical nature, that shows that the use of tables throughout history has supported the gradual development of new epistemological beliefs and has favored conceptual advances in thinking, particularly in Mathematics.

In the case of the experimental sciences, such influence is already present from the proto-scientific stage, even though this opposes them both in a historical and logical sense. In mathematics, however, that proto-scientific stage ends up being part of the discipline (Rashed, 2003). In fact tables, as recent mathematical objects, have experienced an interesting development that we review at the end of the chapter to complete the picture.

For the present case, it is necessary to add to the study of certain para-mathematical notions, tools that are suitable for the advancement of the discipline but not included in it.

Thus, we propose that the contribution of the tables to Mathematics is performed in three different ways: proto-mathematical, para-mathematical, and mathematical. The data that we will provide, that use Chevallard's categories of historical stages in the development of mathematical objects to evidence aspects of interest for teaching tables, will enable us to show elements both of Piaget and Garcia's (1989) genetic epistemology and of factors that help to explain the difficulties of the implicit cognitive process, from the *social interactionist* perspective.

We share the assumption that concept development is not universally consistent through time; also, we do not establish a parallel between different cultures (cf. Schubring, 2011); even so, a review of historical and epistemological character like the one that we will render, may give us some lights on the conceptual evolution of the object table.

According to Piaget and Garcia (1989), the various stages or levels that occur in the construction of knowledge are sequential, and that order is evident in history; these stages are perceptible both in historical processes and in those that arise in learning, and apply both to the history of Mathematics and to the evolution of concepts and their domains and levels of development. However, we share with Schubring (2011) that ruptures and new directions in the history of Mathematics owe much to epistemological changes, which in turn are connected to changes in the systems of scientific activity. In this approach, history is discontinuous and provided of interaction between personal experience and cultural knowledge that enable conceptual development.

Chevallard (1991) defines proto-mathematical and para-mathematical notions regarding teaching and learning as those that are not taught by the teacher, nor directly evaluated, but that, if a student does not have them, he/she cannot (re)build the knowledge and/or use it, and is unable to make real progress. The para-mathematical ones are required as tools (idea of proof, say), to be consciously used as instruments to describe other objects, but they are not considered as an object of study in themselves. The proto-mathematical notions are used to solve problems, but they are not recognized as objects, instrument or tool for this study (notion of simplicity, for example essential to treat various mathematical objects or to recognize patterns, necessary for many others). On the other hand, the mathematical notions are built knowledge objects, teachable and usable at school, and serve to study other mathematical objects.

However, the meaning of a particular mathematical notion is first linked to the field of problems to which it historically responds, but then it is decontextualized from it. An example of this is the notion of *distance*, *d*, which appears in the context of measurement in the Euclidean sense; from a contemporary mathematical perspective, it is a para-mathematical notion. This notion is precisely defined from the fundamental properties that characterize it: it is *defined*, that is, $d\;\;x,x\;=0$; is symmetrical, i. e., $d\;\;x,y\;=d\;\;y,x$; and satisfies the *triangle inequality* $d\;\;x,z\;\leq d\;\;x,y\;+d\;\;y,z$. From this precission, it is now possible to define more broadly a *distance* as a function *d* from a set *E* to the set $\mathbf{R}^{+}{}_{0}$ of nonnegative real numbers, that satisfies the properties above. This in turn allows to collect a number of distances that were not known and, more importantly, to define the general concept of *metric space*: *E* provided with a distance *d*. This transformation from a mathematical concept to a mathematical one allowed a reorganization (of the conceptualization and) of teaching, promoting a change in the conception of geometry and thus the separation of affine and metric properties and, moreover, operating in a domain completely different from its original one (Cf. Chevallard, 1991).

In turn, Brousseau (1986) notes that the status of mathematical concept is given by a mathematical theory that allows to define exactly the structures involved and the properties that are satisfied, and that this is usually preceded by a period when the concept was a familiar, recognized and named object, whose characteristics and properties were studied, but still not mathematized –that is, not theorized nor organized–, and culturally unrecognized.

We will focus on two fundamental mathematical ideas from a perspective both theoretical and practical: number and function.

In Section 2, we begin with a rather general notion of *table*, sufficient for reading the chapter, distinguishing it from the list, of one-dimensional character. We will also use the term *tablet* (cf. Neugebauer & Van Hoesen, 1959) for a device on which inscriptions were made, not necessarily in a tabular format. Then we will go into the tables and their relationship to the emergence of the concept of number and of numbering systems – of course, in connection with other areas of knowledge and everyday life. Later we will focus on the tables and their relation to the genesis and use of the concept of function, inside and outside of mathematics. Then we treat the table as a mathematical object itself, and the algebraic structure recently built upon it. We end with some conclusions.

2. GENERAL NOTION OF TABLE

Before the tables, *lists* were used. They comprise enumeration and/or classification – of things, quantities... – and consider an arrangement in columns (vertical reading) or rows (horizontal reading); they have no header and their components are separated by spaces and/or punctuation. The tables, on the other hand, are two-dimensional, are recognized by the usual segmented rectangular grid of rows and columns, whose cells contain headers and/or values.

The Mesopotamian lists (Robson, 2001) contained units of capacity, mass, length, area; in Mesopotamian tables, instead, there were correspondence of units with abstract numbers.

The first tables started to be used as current notebooks are used today: they served to record everyday life, history, law, accounting, exercise, and so on. Ever since prehistoric times registration in certain tables begins to set defining characteristics of contemporary format: a spatial arrangement in rows and columns, with cells that contain data records.

Among the features of the table are the computational (multiplication tables, *barème*), mnemonic (periodic table of chemical elements) and heuristic (truth tables). They are used today as a tool and/or as an object of knowledge, and are everywhere – sometimes transparent, as in newspapers, statements, invoices, websites, etc.

3. ANCIENT TABLES AND THE NOTION OF NUMBER

In this section we show evidence that tables have been important tools, indispensable from a historical point of view, to the emergence of the concept of number, its writing and its use.

3.1 Prehistoric tables: characteristics and appearance of the numbers

Many civilizations left traces of their legacies in different materials: wood, bark, bone, leather, metals, horn, ostrac, clay, textiles, papyri, stones. We will give a brief description of some of them, particularly relevant to our subject.

The first signs that give evidence of mathematical thinking in these records are from the upper Paleolithic or prior to it.

For at least 37,000 years ago in Africa, counting instruments start being used. The Lebombo bone stands out (Figure 7), with 29 parallel incisions in a column (Boyer and Merzbach, 2010).

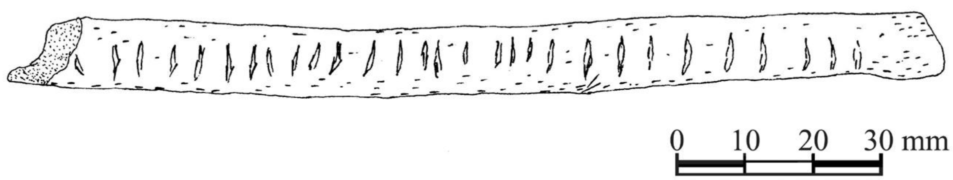


Figure 7. Lebombo Bone (Border Cave, Lebombo Mountains between South Africa and Swaziland)

Collections of records of foremost interest to the mathematician archaeologist have been those that gather together about 30 marks, related to the synodic month. Several traces of this type have been found, but most are still not disclosed academically. For example, decorated bone pin in the Gorge d' Enfer[6], almost 34,000 years old (Figure 8), contains marks grouped into three columns: 31 (= 8 +8 +10 +5) on the central face, 39 (= 9 + 2 +8 +4 +3 +5 +8) in the right column, and 33 (= 3 +2 +5 +10 +5 +8) in the left column.

[6] http://www.britishmuseum.org/explore/highlights/highlight_image.aspx?image=ps255934.jpg&retpage=21072

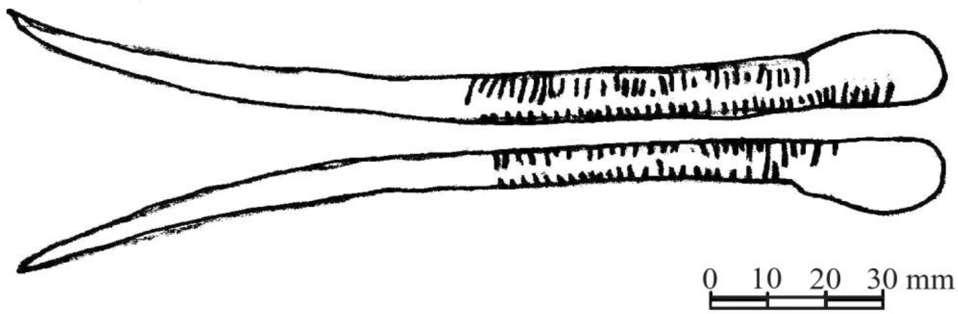


Figure 8. Bone pin in the Gorge d'Enfer (Dordogne, Francia).

A wolf bone found in Moravia is of around 30,000 years B. C. (González et al., 2010), has 55 marks organized in columns, of 25 and 30 marks, respectively, and within each series they are arranged in series of five (see Figure 9).

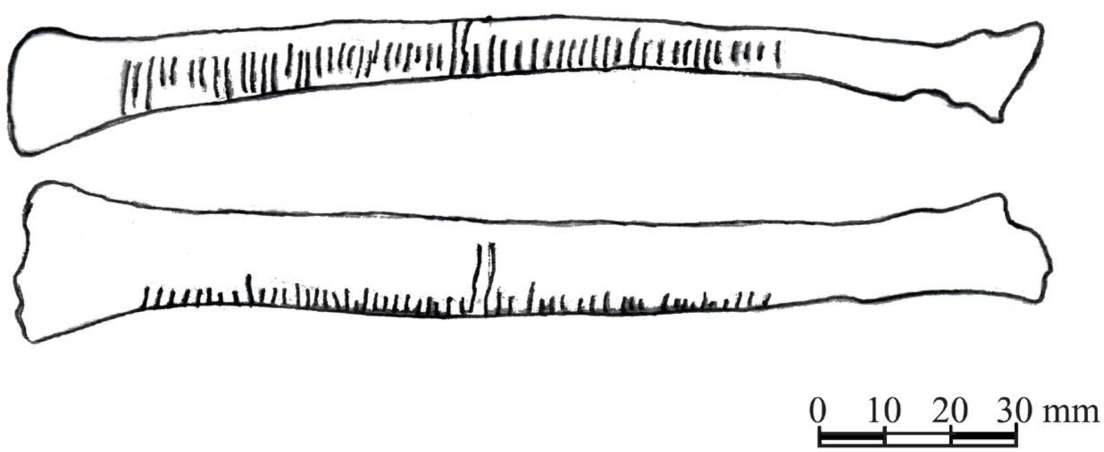


Figure 9. Bone Wolf (Dolni Vestonice, Moravia, Czech Republic).

The Ishango bone, 25,000 years B. C., has 168 incisions along three columns and it would not represent a lunar calendar regularity. Joseph (2011) interpreted it as a proto-writing record of numerical information. In a column there is 11, 13, 17 and 19; in another, 3, 6, 4, 8, 10, 5, 5 and 7; and, finally, 11, 21, 19 and 9 (see Figure 10).

Figure 10. Bone Ishango (border between Congo and Uganda).

In the Brassempouy reindeer antler, in des Landes, France, of about 15,000 years B. C., the marks are composed of groups of 1, 3, 5, 7 and 9 straight lines, respectively (see Figure 11).

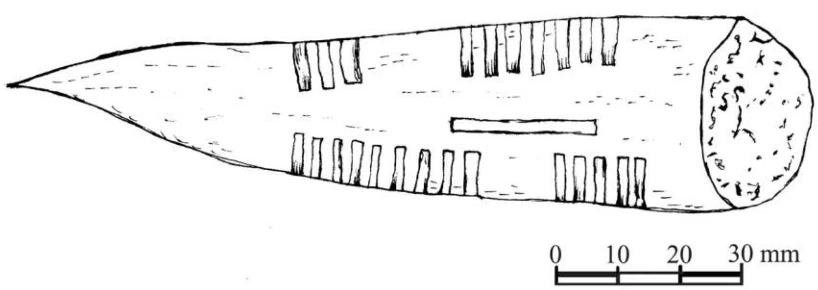


Figure 11. Reindeer antler (Brassempouy, Francia).

In the gradual development of writing highlights the one known as Vinca, that evolves from simple symbols of the seventh millennium and culminates, as recorded in the inscriptions tablets of Tartary, about 5,300 B. C. (see Figure 12). They present an array of four cells with carefully aligned ideographic symbols and layout of divisor segments, which witness the transition from marks to writing, in tables.

Figure 12. Tablet of Tartaria (Cluj-Napoca, Rumania).

Only in 1993, in Greece, the tablet of Dispilio (Figure 13), a Neolithic settlement whose approximate date is 5,260 B. C., was discovered. On its wooden surface there are about 43 signs engraved, arranged in four columns spaced apart; the separation of the data is explicit and thus more complex.

Figure 13. Tablet of Dispilio (Kastoria, Macedonia, Greece).

The tabular formats shown are an example of the evolution that for millennia thinking in relation to the numbers experienced. Through a process of about 300,000 years of cultural development, a gradual notion of number was built (Merzbach and Boyer, 2010).

While the formal concept of number extricates of iconic representations, the one of *numerosity* comes from only one of them: marks, signs that share some characteristics with its referent (Peirce, 1931); for example, the cardinal icons, physical marks on objects – notches in a bone –produce representations that are based on an enumeration of elements: “an object, and another object, and another object” instead of assigning a number to the set “three objects” (Wiese, 2003, p. 386).

Also, language development was essential to the birth of abstract mathematical thinking. At the dawn of the proto-writing, the iconic representations preceded words to represent numbers. According to Wiese (2003), it is language that gives human beings the ability to move from iconic representations, shared with other species, to a generalized concept of number. The concept of cardinal marks based on an object plays a central role in the development of discrete numerical representations, and it is based on this iconic development of number that emerge early lists of consecutive marks.

3.2. Mesopotamian Tables: quantifiers, measures and numbers

For a long period, the scribes of Mesopotamia used almost only lists. Tables appeared over half a millennium after the invention of writing; they were then partially adopted, disappeared and were reinvented several times, to settle only in the XIX century B. C. (Robson, 2001). Hallo (1964) places the transition from lists to tables in the Old Babylonian period.
In the vicinity of the XVIII century, B. C., the powerful potential of the table as a management tool for quantitative data emerged, and it continued to develop gradually but not continuously throughout 500 years, at least in the city of Nippur and its surroundings. Robson (2001) believes that it is no coincidence that the tables arised only after the invention of the sexagesimal positional numerical system and the conceptual separation of quantifier (number) and quantified (object). It is the stage of gradual transition from simple lists, linking number and object, to double lists and then to tables. In the latter the distinction between quantitative

and qualitative is manifested; by means of the physical layout of dividing lines, it is possible to see and explore numerical data and relationships in a way hitherto unimaginable (see Figure 15).

Documents containing tables are mainly in large institutional management files of Sumeria and Babylon, the detritus of the education of scribes, and academic libraries in the great temples (Robson, 2001). The first stage of education began with metrology lists learned by heart, the knowledge of measures being necessary for accounting and administration. Then followed the metrological tables, which contained information related to the number and metrology systems used, similar to lists; they established a relationship between the measures and numbers. The 'curriculum' – focused on the number as measure and counting, and the calculation in the multiplicative field – continued with number tables, with abstract numbers and operations of multiplicative nature (Proust, 2010).

From 10 to 20 % of the tablets found are of mathematical character (Proust, 2010). These jointly used different metrology systems, and exhibited at most two axes of organization (see Figure 14, Proust, 2005): in the horizontal, different types of numerical information are categorized, and, in the vertical, the data of different individuals or areas. Calculation and organization generally go from left to right and top to bottom, in the direction of cuneiform writing (Robson, 2001; Friberg, 2007).

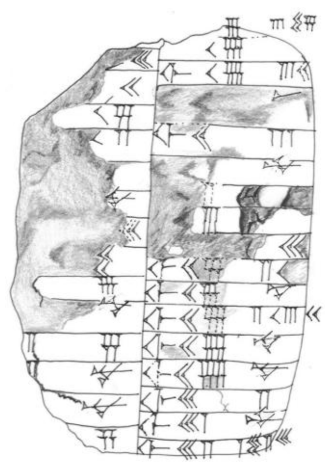
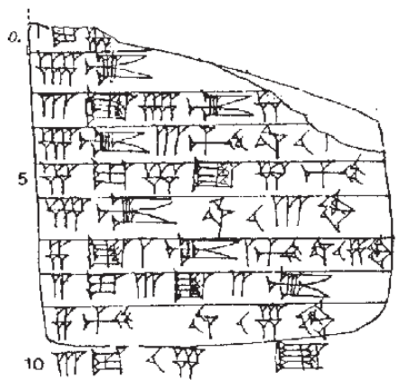


Figure 14. School tablet. Table of inverses Neo-Sumerian of the City Nippur, late 3000 a. C. (Istambul Museum).

The tablet on the left in Figure 15, dated on 2,050 B.C., presents a counting that is not tabular, in which there is no physical separation between numerical and descriptive data, or between different categories of data. In the one on the right in Figure 15, of 2,028 B. C., one can observe a tabulated counting with delimitations and headers: laterals at the far right and in columns at the bottom row (Robson, 2003).



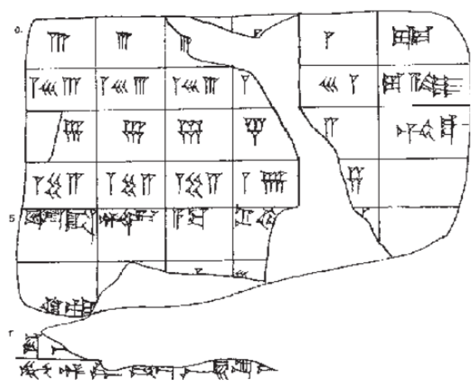


Figure 15. Tablets without table's format (c. 2050 a. C.) and with table's format (c. 2028 a. C.).

On the other hand, lexical lists are Sumerian tablets that provide "a kind of inventory of concepts, a proto-dictionnary..." (Goody, 1977), and in becoming tables they represented a significant change in the modes of thinking in terms of "formal, cognitive and linguistic operations that this new technology of the intellect opened" (Ibid., p. 95). The written list can be read in different directions, has precise beginning and end, a limit. The Babylonians had complex lists, virtually tables, a type of linguistic recoding that activates thought processes; this greatly facilitates the classification of information: in listing, the data are decontextualized from their immediate reality and so its reorganization is made possible; the list increases visibility and definition of classes, facilitates the hierarchical order in societies with writing (Ibid.). The table is a means for ordering our knowledge of the classification schemes, symbolic systems and ways of thought (Goody, 1976).

The first mathematical data table in history is of 2,600 B. C., it comes from Shuruppag (see Figure 16); it has three rows with ten columns. The first two columns refer to the list of measures of length in descending order in *rod*[7] (from *c.* 3600 to 360 m in the contemporaneous system) and the last column contains the square area (Campbell-Kelly et al., 2003).

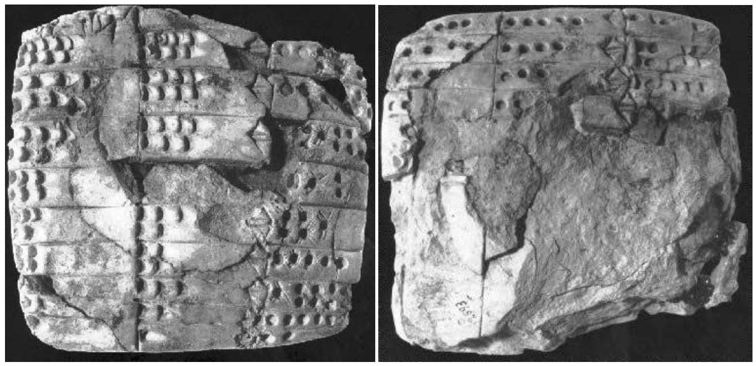


Figure 16 Shuruppag Table, front and reverse, 2,600 B.C. (Vorderasiatisches Museum, Berlin).

The well known Plimpton 322 tablet (see Figure 17), of 1,800 B. C., is a table of measures 13x9x2 cm, inscribed on one side only; it contains words and numbers, in 4 columns and 15 rows, which have been interpreted as Pythagorean triplets and also as reciprocal pairs (Proust, 2010; Friberg, 2007). Robson (2002) argues that historical documents can only be understood in their historical context, and therefore considers that this table, whose content was school mathematics and its role to help the teacher, in a school environment for Mesopotamian scribes.

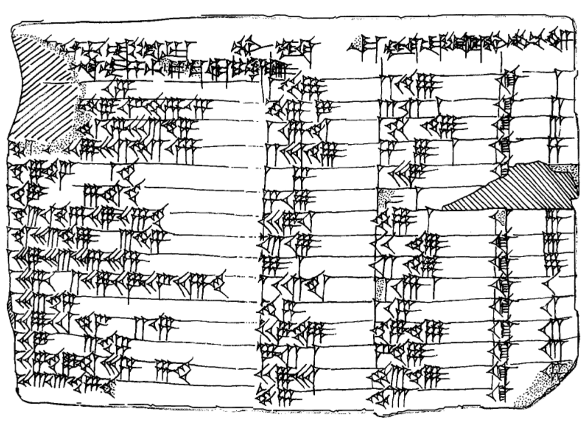


*Figure 17 P*limpton 322, c. 1,800 B.C. (University of Columbia Rare Book and Manuscript Library, New York).

[7] 1rod = 6 m

3.3. Egyptian Tables: fractions and astronomy

The graphs and numerical tables in Egypt belong to formats that order information in rows and columns, and whose use was to generate information derived from the processing elements of a row or column (Ross, 2011).

The two tablets of Akhmim, or of Cairo, of 2,000 B. C., are inscribed on both sides, contain lists and some numerical problems of equivalence of measures of capacity. They present some miscalculations but the importance lies in that the system of Egyptian fractions might have originated in trying to divide grains units into smaller ones (cf. Gardner, 2012).

Stobart tablets, *c.* 100 A. D., are four tablets covered with plaster, with three holes to bring them together as a book (see Figure 18). Each side of each one has five columns, separated from each other by red lines, of about 30 rows; the horizontal grouped data of one planet each year. They are written in Demotic and represent annual records of the movements of five planets (Neugebauer and Van Hoesen, 1959).

Figure 18 Stobart Astronomical Tablets (Liverpool Museum).

In the table in Carlsberg Papyrus 32, of II century A. D., there is a column with the daily motion of Mercury *A* and another with its displacement *B* relative to its maximum elongation. It states $A_n = n \cdot v$, (*v* the observed numerical value), and $B_n = B_{n-1} + A_n$; Thus, all values are generated from a single parameter (Ross, 2011), and the table serves to organize and display information. However, in general, the ordering of planets and astronomical events of the Egyptians is closer to the list gender.

3.4. Maya Tables: calendar and numbers

After the conquest of America, the Mayans were devastated, and although there is little material collected, it has been inquiring about your math.

The Mayans came from a civilization that inhabited Mesoamerica since 2,000 B. C., but its classical period ranges from 250 to 900 A. D. Their mathematics is registered in hieroglyphic inscriptions in columns known as *stelae* (date of construction, major events in 20 years, names of personalities) paintings and hieroglyphics on walls of caves and mines (daily life, scientific activities), and codices (see Figure 19). Of the latter outstand the Dresde's, the Peresianus and the Tro-Cortesian, arranged in long strips of bark or leather folded lengthwise.

Figure 19 Mayan Codex.

The Dresden Codex is a copy made in leaves in s. XI A. D. of the original work 3 or 4 centuries before, corresponds to 78 pages on 39 leaves. It is one of the main sources of information about the numbering system and Mayan astronomy. It consists of 10 chapters, of which the numbered 3, 4, 5 and 10 have tables (cycles of: Venus, solar and lunar eclipses, rainfall, and Mars; and multiplication). It contains a calendar more precise than the European at those times then (Joseph, 2011), and also one of Venus; in this last, each sign represents the day on which a particular position of one of the five periods has begun. In addition, the tabular arrangement relates to how the scribes recorded numbers: positional system and vertically oriented, top-down writing. Numbers arranged in rows and columns are observed; and it is possible to read 9, 9, 16, 0, 0 – i. e., not a pure vigesimal system (cf. Boyer, 1991).

In many of these written forms, numerical objects represent periods associated with dates, usually cultural or astronomical cycles (phases of Venus, lunar or solar eclipses). The Maya developed the measurement of time more than the metrological of other civilizations. Cauty (2006) notes that often a particular reference date was set; the scribes have sought invariants of translation operators and worked the numbers as instruments.

The Mayans solved all their calculation problems with only with multiples' tables, and of invariant dates, and marked distinctions between ordinal and cardinal numbers; thus, zero had a different symbol for its cardinal ordinal use. The ordinal was used in the departure or arrival of a cycle, the date in a temporal sense, and the cardinal was used in calculating durations (Cauty and Hoppan, 2007).

3.5. Andean cultures: the *quipu*

In considering tables in their role of repositories, we must consider in this gathering of backgrounds the *quipu*. Several of our cultures of South America used a system of accounting records known as *quipukuna* (plural of Quechua quipu) that consisted of a mnemonic system of strings of one or more colors.

The chronicler Garcilaso de la Vega (1609), in Chapter VIII of his book "*Comentarios reales de los Incas*" entitled "They counted by yarns and knots; there was a high fidelity among counters", precises the use and meaning of this system:

> "Quipu means knot and to knot, and is also taken for account, because the knots gave it of anything. The Indians did yarns of different colors: some were of a single color, others of two colors, others of three and others of more, because the colors simple, and mixed, all had their significance by themselves, the threads were very crooked, three or four thread, and thick as an iron spindle, and long at three quarters of a rod, which strung in other wire by their order in the long, by way of fringes. By the colors they drew what was contained in that thread, such as the gold by the yellow color, and silver by the white, and by the red people of war."

Garcilaso also reports the number system used, which included zero:

> “The knots were given in order of unit, ten, hundred, thousand, ten thousand, and seldom or never went to the hundred thousand, because, as every village had its account of itself and each metropolis the one of his district, never got the number of these or those such amount as to pass the hundred thousand, that in the numbers down below there was enough.”

Later, he draws attention to knots builders,

> “At the top of the threads they put the greatest number, which was the ten thousand, and below a thousand, and so on until the unit. The knots of each number and each thread were evenly matched with each other, neither more nor less than a good accountant puts for a large sum. Of these knots or quipus some Indians were in charge, and they were called *quipucamayu*: it means, who is in charge of the accounts, and although at that time there was little difference in the Indians from good to bad, who, by the little malice and good governance that had everyone could be called good, nevertheless they chose for this trade and to any other the most approved and those who had been longest given experience of their goodness. They were not giving as a favor, because among those Indians it never was used alien favor, but only their own virtue.”

These builders and managers of *quipu*, the *quipucamayus*, adds the chronicler, existed in every town, although it were small, four to thirty people,

> “…all had the same records, and, although all records being the same, it was enough to have only one accountant or clerk, the Incas wanted to have many in every town and in every faculty, by excusing the falsehood that could be among the few, and they said that having many, all had to be in wickedness, or none.”

In addition, the chronicler, in Chapter IX, clarify the work of *quipucamayus* in *quipu*, in recording and reading of data of interest to the government of the empire:

> "They settled for their knots all gave tribute each year to the Inca, putting everything by their genera, species and qualities. They settled people going to war, dying in it, those born and dying each year, for their months. In short, we say that they wrote in those knots all the things that were to account in numbers [...].”

Figures 20 and 21 show some of the drawings by Guaman Poma in the early seventeenth century, on the *quipu* and some *quipucamayus* obtained from the book "The first new chronicle and good governance", written by Don Phelipe Guaman Poma de Aiala (1615).

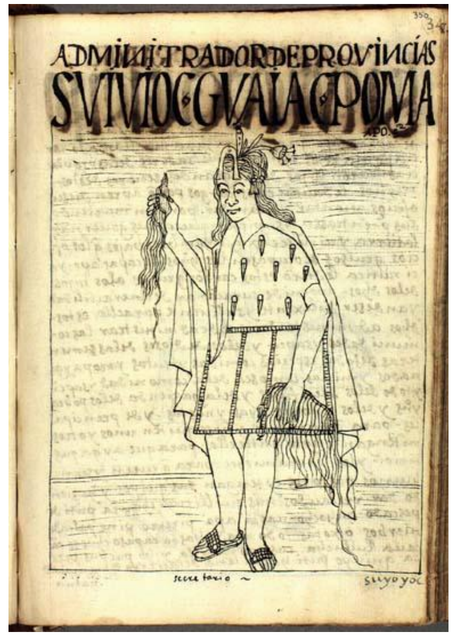


Figure 21 Secretary-Accountant and Provinces Manager (Guaman Poma, 165, p. 348 y 358).

Figure 20 Incan storage manager, and Accountant Major and Treasurer with *quipu* and *yupana* (Guaman Poma, 1615, p. 335 y 360).

4. TABLES AND THE CONCEPT OF FUNCTION

As we have been describing, the use of tables is closely related to the genesis and development of the concept of number, but clearly they do not take each number separately, but they explicitly associate to it another number, or a certain amount of mass or volume, etc.
More generally, a common table assigns, to some objects, other objects. Whether a registration grain astronomical data or numbers, to a given position (in a column) is associated a (single) data (in another column) or a data set (in several parallel columns, as in the possible Pythagorean triples already mentioned, e. g.).

The idea of function is seen there: a map that assigns to each element x of a set *A* and a single element of a set *B* (from lists of data, e. g.); such correspondence does not always require a formula.

In ancient times, a function can be recognized in a table, subject to the condition that the corresponding rule is defined and that giving a value to x is adequate to find y. We can recognize a proto-mathematical notion of function in the rules for measuring areas of geometric figures, and in tables that are used to facilitate early complicated calculations.
More generally, the mathematical concept of function had to move from a para-mathematical concept to a proto-mathematical one, and decant, through reflection and algebraic work, in a proper mathematical concept.
Moreover, the notion of function appears expressed, more or less accurately, in various tabular, graphical and algebraic records.

The ubiquity of the idea of function makes it difficult to separate some of its elements in order to expose its historical development. Here we will just show how the genesis of the concept of function is inextricably linked to the concept of table.
To avoid ambiguities, we include a formal description of binary relations and functions – omissible, provided that we identify a function with the set of pairs of objects that it determines.

4.1. The function concept

Hereinafter, we use the symbols and terminology in Mena-Lorca (2004).
If A, B are sets, a (binary) relation R from A to B is a correspondence xRy between elements $x\ of\ A\ and\ y\ of\ B$ in this case is said that y is an *image* of x, and that x is a *preimage* of y. The domain Dom(R) of R is the set of preimages of R.
A function from A to B is a relation that associates each element of A a unique element of B:

i. $(\forall x \in A)(\exists y \in B)(xRy)$
ii. $(\forall x \in A)(\forall y \in B)(\forall y' \in B)(xRy \wedge xRy' \Rightarrow y = y')$

It we have ii., we say that R is functional (in y). If so, if we have not i., we can still have a defined function from $Dom\ R$.

It is customary to identify the relationship with its graphic $G_R = \ x, y \ \in A \times B : xRy$: rather than think about the type of relationship established between objects – y is the square of x, y is brother of x, it is considered the set of pairs of elements it determines – (3, 9)...; (John, Peter)....

The role of tables in the developing of the concept of function is quite natural: they point out, as we shall see, maps and the pairs that are formed, with the restriction that, if the function is between large sets, the table only provides a sample, that should be extended and/or interpolate.

4.2. Tables and functions in Mesopotamia

There are opposing views regarding the development of the concept of mathematical function in the Mesopotamian civilizations. Pedersen (1974) states that these had a certain sense of functionality, but Youschkevich (1976) estimates that in Antiquity there was not a general idea of a functional relation, as the study of particular cases of dependencies between two quantities had not yet isolated from the general notions of variable quantities and functions.
In fact, Babylonian mathematicians were not limited to a simple tabulation of empirical data but in finding regularities, both linear interpolations performed as geometric, tabulating

functions corresponding one, two and even three variables. In any case, they have a protomathematical idea of correspondence.

4.3. Hellas: interpolation and infinitesimals

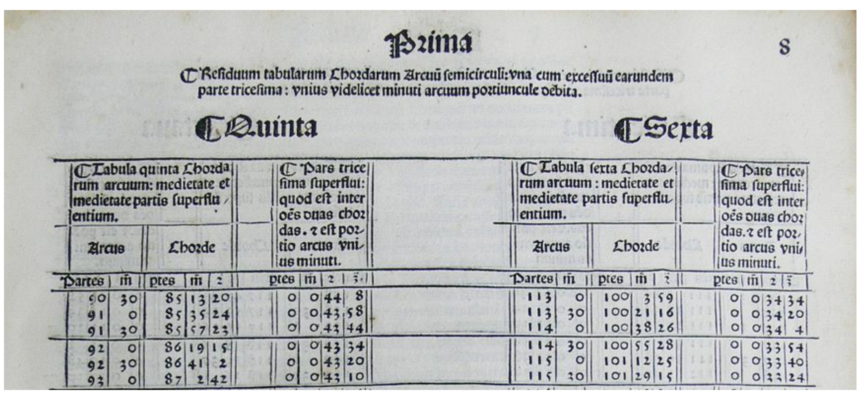
Definitely an idea similar to the notion of function had been developed in Greece and nations under their influence, since Hipparchus lunulas.

Youschkevich (1976) notes that, in ancient Greece, the functions relating to mathematical and astronomical problems were treated mathematically. They were tabulated using linear interpolation, and in the simplest cases, the ratio of two infinitely small quantities was found. According to Sidoli (2009), no tables have been found in the texts of the first Hellenic mathematicians, although he notes the sieve of Eratosthenes, there is evidence only in papyri and numerical tabulation epigraphs (tables fractions and *Keskintos*: a carved stone that contains a table with estimates of movements of planets, 100 B. C.).

Ptolemy, circa 150 A. D., in his *Almagest* describes the geocentric system and the apparent movement of stars and planets. The mathematical tables that he uses are sets of numerical values of lengths and arcs obtained from geometric patterns, from geometric objects with given numerical values, or from a geometric pattern with astronomically determined parameters (see Figures 22 and 23). They can be considered a quantitative representation of the model, and also a tool to evaluate specific values for the model (Sidoli, 2011; Aaboe, 1964). To Pedersen (1974), in the Almagest is the first trigonometric table of History, with data and calculations with 360 rows and three columns.

Studies of Sidoli's (2011) on the Almagest show seven types of texts; among them the Table type (list of numerical values derived from the model and generally for use in subsequent calculations) and the Algorithm type (description of how can we use the values in the table for some calculations, for certain problems or for the apparent positions of the stars). Such tables do not work independently, but rather work with other types of text forming a "tabular nexus": given the geometric properties of the model and some parameters, the numbers in the table are determined; a table or series of tables provide a numerical representation of all key components of the geometric model; an algorithm describes how the different table entries can be used to calculate the phenomena that are really seen.

Figure 22 Astronomy tables from Ptolemy's Almagest (Arabic translation of Ishaq b. Hunayn (830-910), revised by Thabit b. Qurra (836-901). Copied by Ibrahim ibn Muhammad al-Sharfi, Maghreb in Spain, 1221. Manuscript on paper (131 sheets; 25.5 x 18.5 cm).



Prima 8

Residuum tabularum Chordarum Arcuũ semicirculi: vna cum excessuũ earundem parte tricesima: vnius videlicet minuti arcuum portiuncule debita.

Quinta

Tabula quinta Chordarum arcuum: medietate et medietate partis superfluentium. Arcus: Partes	m̃	Chorde: ptes	m̃	2	Pars tricesima superflui: quod est inter oẽs duas chordas. et est portio arcus vnius minuti. ptes	m̃	2	3
90	30	85	13	20	0	0	44	8
91	0	85	35	24	0	0	43	58
91	30	85	57	23	0	0	43	44
92	0	86	19	15	0	0	43	34
92	30	86	41	2	0	0	43	20
93	0	87	2	42	0	0	43	10

Sexta

Tabula sexta Chordarum arcuum: medietate et medietate partis superfluentium. Arcus: Partes	m̃	Chorde: ptes	m̃	[illegible]	Pars tricesima superflui: quod est inter oẽs duas chordas. et est portio arcus vnius minuti. ptes	m̃	2	[illegible]
113	0	100	3	59	0	0	34	34
113	30	100	21	16	0	0	34	20
114	0	100	38	26	0	0	34	4
114	30	100	55	28	0	0	33	54
115	0	101	12	25	0	0	33	40
115	30	101	29	15	0	0	33	24

Figure 23 Extract a table of Ptolemy's Almagest (latin edition, Venecia, 1515).

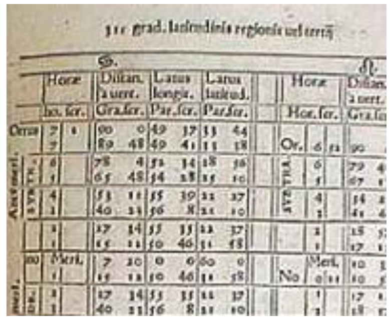
Sidoli (2009) notes that the tables of Ptolemy are tools while objects, that is, they facilitate and organize the calculation in order to reveal underlying patterns, made evident by the numbers. He adds that the texts by Hipparchus, Menelaus and Diodorus would contain mathematical tables of a similar kind: tables to catalog quantitative information, to facilitate and/or enable the calculation and develop an algorithm (moving between columns) to represent the underlying regularity between two (or more) magnitudes.

4.4. European tables and astronomy

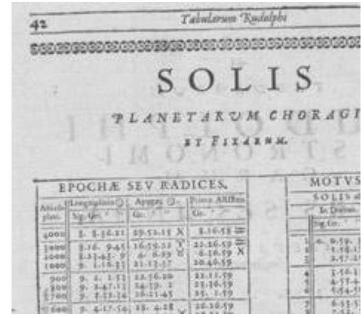
After the fall of the old society, science had already emerged in countries of Arabic culture. This increases the number of functions used, as trigonometric ones, and methods for tabulation are improved: it began by using linear and quadratic interpolation, and progress was made in the study of positive roots of cubic polynomials, by conic sections (Youschkevich, 1976).
The geocentric model was imposed on the heliocentric of the Pythagoreans and of Aristarchus of Samos (310 B. C.). About the geocentric model were constructed the so called Alphonsine Tables (1252) to provide an outline of practical use to calculate the position of the sun, moon and planets. They influenced to the Renaissance and were very useful for geography, helping to localize the terrestrial coordinates, and also for navigation, by supplying orientation by means of constellations and planets.
Copernicus proposed that the appearances in favor of the geocentric model are also consistent with the heliocentric. Its design, with modifications, will be fully accepted only from the development of Kepler's and Galileo's. Supported by an extensive database tabulated for decades by his mentor Tycho Brahe, Kepler consolidates his three laws of planetary motion.

Brahe was as user as producer of astronomical tables, and is considered the last of the great astronomers before the telescope. In 1563 a conjunction of planets predicted by existing tables occurred, but Brahe observes that all predictions about the date of the conjunction were wrong, and realizes the need to compile new and accurate planetary observations that allow him to build more accurate tables.
When Kepler succeeds Brahe as imperial mathematician in Prague, is based on Brahe's data to complete his own calculations, and publishes the Rudolphine Tables (1627), exceeding the inaccuracies of the Alphonsine and Prussian tables (developed by Reinhold in 1551), and containing the positions of about a thousand stars measured by Brahe, more than 400 over Ptolemy (Figure 24).

Alphonsine tables
Tabulae alphonsinae (1545)



Prussian Tables
Tabulae prutenicae (1551)



Rudolphine Tables *Tabulae rudolphinae* (1627)

Figure 24 Astronomical tables constructed according to the Ptolemaic system.

4.5. Tables and expeditious calculations

The advent of printing had facilitated the creation and dissemination of written works.
At a time when, with the exception of abaci, there were no mechanical calculating devices, the table offered speediness to compute. Napier's logarithms abbreviated, facilitated and precised calculations of triangles and figures; with them more promptness, security and accuracy were obtained. Napier (1614) includes in his first book 90 tables of sines and cosines with their logarithms. Promptly, Gunter (1617) and Briggs (1633) published the first tables of logarithms

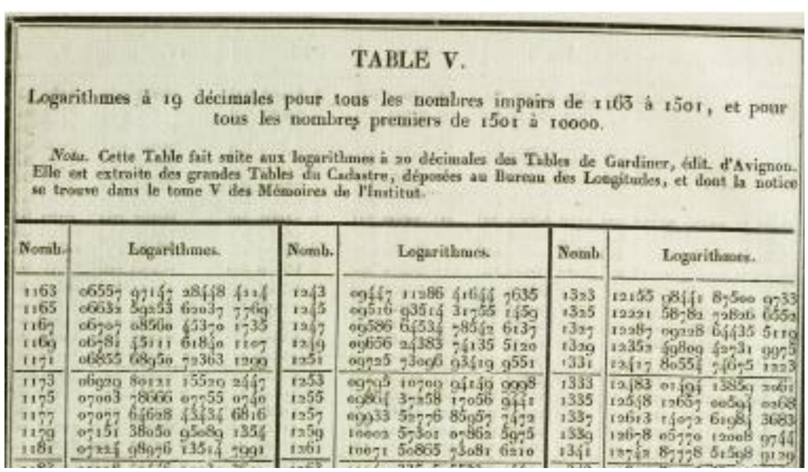


TABLE V.

Logarithmes à 19 décimales pour tous les nombres impairs de 1163 à 1501, et pour tous les nombres premiers de 1501 à 10000.

Nota. Cette Table fait suite aux logarithmes à 20 décimales des Tables de Gardiner, édit. d'Avignon. Elle est extraite des grandes Tables du Cadastre, déposées au Bureau des Longitudes, et dont la notice se trouve dans le tome V des Mémoires de l'Institut.

Figure 25 An extract of a table of logarithms published by Legendre (1826, Table V, page 260). There are more than 20 errors in this table built for 120 numbers, especially in the last digit. (Image obtained from Roetgen, 2011).

of trigonometric functions, and began a profusion of tables derived from those (Roegel, 2010). Efforts made to reach agreement in an international metric system also promoted the development of conversion tables. Among others, Barrême in the seventeenth century wrote several books of practical mathematical tables, the barème, that exempted the public the task of making these calculations; they were reprinted many times, even in pocket size.
Lardner (1834), states that the British parliament elected those who would have the honor to make the very useful lunar tables, which facilitated navigation and nautical astronomy calculations. The first, Mayer, using a formula of Euler’s published his tables in 1766; Mason replaced them in 1777; Burg published his in 1806 and uses the theory of Laplace; in 1812, Burckhardt’s appear more accurate. Others continue building these tables. Parallel working committees evaluated scientific knowledge to renovate and/or build more complete lunar tables.

Charles Babbage, in a paper presented at the Astronomical Society of London in 1829, shows some common mistakes in many tables of logarithms (an example of this in on the table is shown in Figure 25). Lardner, warning the public of the existence of errors in manual tables, starts promoting the potential of calculating machines, such as the analytical one invented by Babbage. The firsts to build a machine capable of producing printed mathematical tables were Georg and Edvard Scheutz, who based on Babbage’s design. In 1849, the first table automatically made in the first calculating machine that prints (Merzbach, 1977) appears. Passing to the automated tables spread the use and even more accuracy and quickness is gained.

4.5.1 Expedited calculations provided by Chilean Ramón Picarte

The Mathematical Tables Committee of the British Association for the Advancement of Science, later known as The Royal Society, led the activity of tables construction for almost a century, from 1871-1965. Croarken and Campbell-Kelly (2000) note that, during this period, the construction of a table went from being a private, solitary activity to that of a group of organized people that calculated and used computing machines. Its best known product was the Mathematical Tables Series, synonymous with precision and perfection in typography. After World War II, electronic computers would take the role of these table builders.

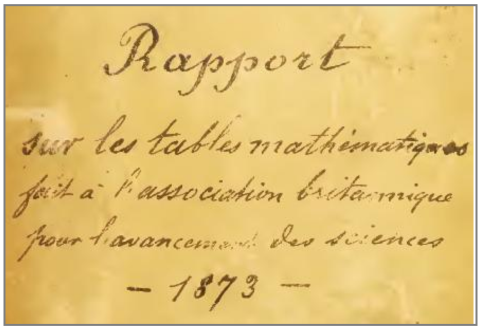
A few years before the aforementioned Committee, and as stated in the Annals of the University of Chile, to do science in Chile and provide a speedier type of tables[8] was an arduous task for Chilean Ramón Picarte Mujica:

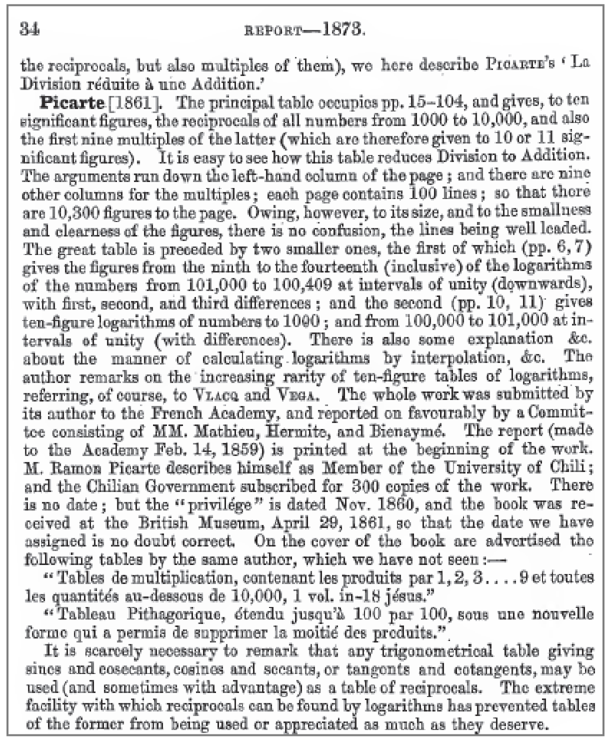
"Due to vigils and contraction to this science, he came to invent a division Table that reduced this long and painful operation to a simple sum. [...] Caught by enthusiasm with this finding, he communicated it to competent persons and friends, believing that they would feel the same satisfaction [...] Everywhere, Picarte found nothing but contempt, indifference, and at best, some compassion. Of many proceedings, he only managed to get that all should think of him as mad, and as such they spoke to him whenever Picarte played the idea that bore him so worried." (Gutiérrez y Gutiérrez, 2000, p.8).

As can be seen in Figure 26, the work of Ramón Picarte is included in the 1873 Report of the Committee, intended to present the *state of the science* of the time being. In that book, section 7 addresses the issue of Ramon Picarte's reciprocals table; the authors detail the construction, characteristics and length of the tables, in addition to favorably comment on the use of it.

Ramón Picarte is considered the first scientist born and educated in Chile that successfully managed to publish his work abroad. Born in 1830, when the development and teaching of Mathematics in Chile newly started (Gutiérrez and Gutiérrez, 2000). At that time, there were about 50 thousand inhabitants in Santiago and only 31 "first letters" schools, to which 1733 students attended, who were taught numeracy based on the four basic operations.

[8] Anales de la Universidad de Chile, 1858, Tomo XVI, pp. 67-74.





34 REPORT—1873.

the reciprocals, but also multiples of them), we here describe PICARTE's 'La Division réduite à une Addition.'

Picarte [1861]. The principal table occupies pp. 15–104, and gives, to ten significant figures, the reciprocals of all numbers from 1000 to 10,000, and also the first nine multiples of the latter (which are therefore given to 10 or 11 significant figures). It is easy to see how this table reduces Division to Addition. The arguments run down the left-hand column of the page; and there are nine other columns for the multiples; each page contains 100 lines; so that there are 10,300 figures to the page. Owing, however, to its size, and to the smallness and clearness of the figures, there is no confusion, the lines being well leaded. The great table is preceded by two smaller ones, the first of which (pp. 6, 7) gives the figures from the ninth to the fourteenth (inclusive) of the logarithms of the numbers from 101,000 to 100,409 at intervals of unity (downwards), with first, second, and third differences; and the second (pp. 10, 11) gives ten-figure logarithms of numbers to 1090; and from 100,000 to 101,000 at intervals of unity (with differences). There is also some explanation &c. about the manner of calculating logarithms by interpolation, &c. The author remarks on the increasing rarity of ten-figure tables of logarithms, referring, of course, to VLACQ and VEGA. The whole work was submitted by its author to the French Academy, and reported on favourably by a Committee consisting of MM. Mathieu, Hermite, and Bienaymé. The report (made to the Academy Feb. 14, 1859) is printed at the beginning of the work. M. Ramon Picarte describes himself as Member of the University of Chili; and the Chilian Government subscribed for 300 copies of the work. There is no date; but the "privilége" is dated Nov. 1860, and the book was received at the British Museum, April 29, 1861, so that the date we have assigned is no doubt correct. On the cover of the book are advertised the following tables by the same author, which we have not seen:—

"Tables de multiplication, contenant les produits par 1, 2, 3 . . . . 9 et toutes les quantités au-dessous de 10,000, 1 vol. in-18 jésus."

"Tableau Pithagorique, étendu jusqu'à 100 par 100, sous une nouvelle forme qui a permis de supprimer la moitié des produits."

It is scarcely necessary to remark that any trigonometrical table giving sines and cosecants, cosines and secants, or tangents and cotangents, may be used (and sometimes with advantage) as a table of reciprocals. The extreme facility with which reciprocals can be found by logarithms has prevented tables of the former from being used or appreciated as much as they deserve.

Figure 26. Ramón Picarte tables described in the British Science Report, 1873, p. 176.

Until late 1856, Picarte was a lecturer of Mathematics at the Military Academy. It was a scarce books environment; one example is that the same year Domeyko proposed the first subscription to a mathematical journal: the *Journal de Liouville* (current *Journal de Mathématiques Pures et Appliquées*).

Mathematical tables, in their various expressions, represented a tool commonly used for various activities, such as trade, surveying and navigation (tables for multiplying, dividing, of interest for currency changes, etc.). Picarte was interested in designing and calculating a mathematical table that required great strength in this science, such as logarithm tables or difficult to compute functions.

Gutiérrez and Gutiérrez (2000) indicate that at a time when Lalande tables were the most popular existing in the world, and were in the pocket of every surveyor, navigator or engineer, Picarte invents a table superior even to Barlow's and Goodwyn's tables. Figure 27 shows the top of Picarte's the table of reciprocals, in these he reduced the division to an addition (*sic*); moreover, it was also a table designed to provide speed and accuracy in the calculation, since it delivered 10 significant digits.

In Chile, as a new country, without scientific tradition, with a shortage of libraries and mathematical works, his creation was barely considered. Picarte tried to seek funding for publication but got no support. In early 1857, with no money for travel or contacts, he travels to Europe to present his mathematical studies to the Academy of Sciences in Paris. Two years later, on March 6th, 1859, a French newspaper stated "A young mathematician from Santiago de Chile, Ramon Picarte, not long ago left his country and crossed the seas to address the scales of the Institute [the Academy of Sciences]. His courage and perseverance obtained a good precious reward at the judgment that it was formed at the meeting of the Academy of last February 15th, in which he received the thanks of Academics that simultaneously encouraged him to publish his works". The Academy report was signed by the famous mathematicians of that time Mathieu, Hermite and Bienaymé.

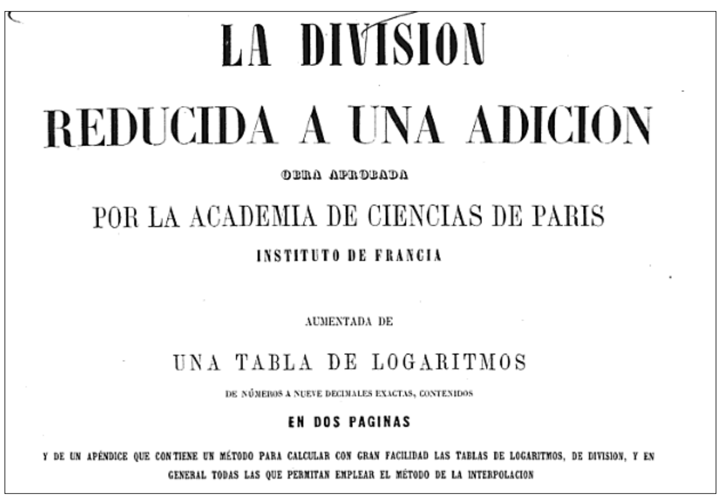

LA DIVISION
REDUCIDA A UNA ADICION
OBRA APROBADA
POR LA ACADEMIA DE CIENCIAS DE PARIS
INSTITUTO DE FRANCIA
AUMENTADA DE
UNA TABLA DE LOGARITMOS
DE NÚMEROS A NUEVE DECIMALES EXACTAS, CONTENIDOS
EN DOS PAGINAS
Y DE UN APÉNDICE QUE CONTIENE UN MÉTODO PARA CALCULAR CON GRAN FACILIDAD LAS TABLAS DE LOGARITMOS, DE DIVISION, Y EN GENERAL TODAS LAS QUE PERMITAN EMPLEAR EL MÉTODO DE LA INTERPOLACION

Figure 27. Tables of the chilean Ramón Picarte approved by the Academy of Sciences in Paris in 1859.

In Paris, the *Monitor de la Flotte* of March 6th reported on Picarte's trip and made a summary of his work, explained what the Tables were and noted that they were superior to the famous Barlow tables reprinted several times in England: "This result is a step given in science. Mathematicians, engineers, industrialists, financiers and the merchants will find it a relief to their long calculations, and above all, make sure not to make mistakes." The *Monitor* ended praising Picarte and the virtues of his work. The *Monitor Universal* and the *Eco Hispano* expressed themselves in similar terms. The latter claimed that analogously "has expressed itself every major newspaper in Paris."" The same welcome to Picarte's prowess was given by Panama's *La Estrella* and other newspapers of America. His tables were sold in France, England, Portugal, Belgium, the Netherlands, Peru and other countries with which France had commercial treaties.

4.6. Various purposes of the Tables

It would be difficult to completely investigate the using of tables in various disciplines. Here we review only a couple of additional examples, illustrative of their importance to the advancement of science.

A new use of the table is introduced by John Graunt by creating life tables in 1662. Graunt used census records tables to analyze and establish classifications of causes of death and create the first tables with chance of life; he sought differences in the numbers using knowledge of the context. Graunt tables arise from a table-based data model to predict and assist in policy-making. In the same period William Petty and Hermann Conring had similar ways of thinking, probably accompanied by studies of data in tables.

Probabilistic thinkers had applied the nascent theory of probability to Graunt tables: Christiaan Huygens and his brother in terms of gambling issues calculated mortality; similarly, de Witt calculates the value of annuities; Halley will build a mortality table that allows the empirical calculation of future life *chances* and annuities; contributions in the same vein were reported by Leibniz and Jakob Bernoulli (Kendall, 1970; Rivadulla, 1991).

The *nomogram* or chart table is a two-dimensional diagram that allows graphical and approximate computation of a function of any number of variables. They were military engineers and other officials in charge of solving quantitative problems of iterative character who sought these aids for calculation (Tournes, 2000). Nomograms had a role analogous to (numerical) tables used for the numerical computation; they offer easier visual interpolation, but they are less accurate. Pouchet (1748-1809) included in his *Métrologie terrestre* an appendix called *Arithmétique linéaire* first attempt to build graphical double entry table. A known nomogram of Lalanne's, dated 1843, used double entry graphs, which he called abacus, as shown in Figure 28.

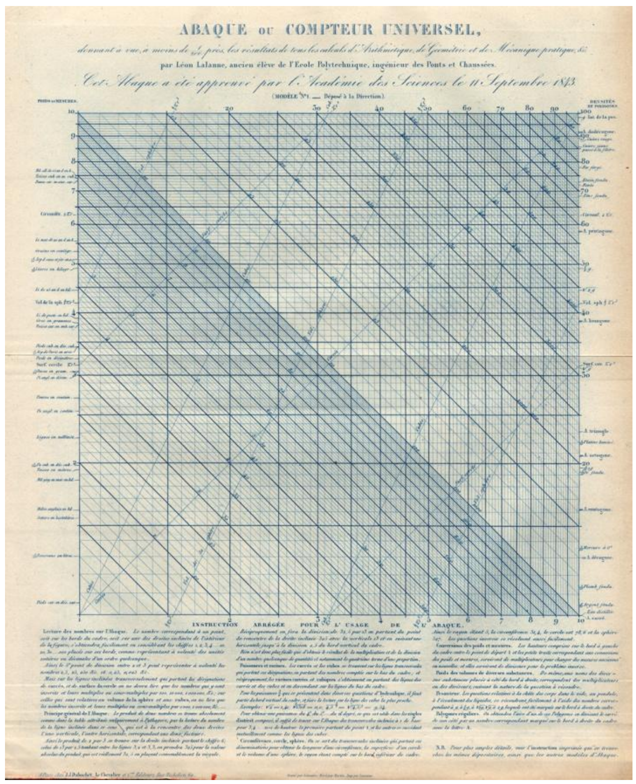


Figure 28 . Abacus of Lalanne, with instructions for use (1843).

Regarding the diversity of purposes of tables, Bertoloni (2004), in studying the role of numerical tables in Galileo and Mersenne, notes that they suit different purposes. A type of tables presents empirical data without theory (weights of materials that are not related to predictions and calculations done within a theory). Another type relates observed data with theoretical implications (analysis of positions of 1572 novas made by Galileo, or trigonometric functions in astronomy). A final type would have "didactic, philosophical and aesthetic purposes" (Ibid, p 188): in some Mersenne tables it is not intended to facilitate the calculation but to highlight the symmetry and regularity of certain phenomena (falling bodies) or to find a height from the time of the fall, or invite us to reflect on the regularities of nature.

4.7. Tables and the formal concept of function

In the late sixteenth century, Leibniz introduces the name *function* but is not he who set the modern functional notation. Euler (1748/1988) puts functions at the center of his treatise *Introductio in analysin infinitorum*, and explains what concerns to variables and functions of these variables, and with it, the notion of function becomes a fundamental idea (Cf. Merzbach and Boyer, 2010); this required the passage to the notion of function that we have given in 4.1, using formulas and equations.
Such transit has been explained in several ways.

Youschkevich (1976) states that, until the beginning of XVI century, functions were introduced only by the old methods; for example, J. Burgi had calculated their logarithmic tables (1620) starting with the ratio – already known to Archimedes – between the geometric progression of the powers of any quantity $q, q^2, q^3, \ldots$ and the arithmetic progression of its exponents. Burgi, in making interpolations, showed to have it understood that this relationship was continuous. However, already Napier would have considered it in his work.

Functional relationships were not only used in Greek mathematics, but also by the Babylonians (Schubring, 2005). On the other hand, Wussing (1998) notes that while Mesopotamian tables or calculation of the Almagest manifest earlier stages in the formation of the concept of function, this requires the passage from a magnitude mathematics considered statically to one of variables, transition expressly made Vieta, Fermat and Descartes in the sixteenth and seventeenth centuries.
On the general notion of function, Tournès (2011, p. 6) states that "... a real 'tabular function concept'' is actually present throughout the history of mathematical analysis."

5. MATHEMATICAL TABLES

As noted earlier, in Antiquity the table was mainly used in presentation in multidimensional arrays and application of interpolation techniques. Here we define the current concept of table and give elements of the algebraic structure constructible from it.

Computer *tabular expressions* are mathematical expressions in tabular format, and have led to generalize the two-dimensional tables. Such expressions were used in the seventies to document requirements of manufacturing an aircraft and since to document and analyze software systems. They contribute to make mathematical notation simpler and intuitively understandable, and are very useful in testing and verification (Parnas, 1991). Parnas (Ibid.) breaks down the information in a software document in mathematical expressions, which he organizes in tables. These expressions, which are mathematical relationships within tables, allow the systematic inspection and mathematical verification of the system.

From motivations such as the above, the table is defined as a mathematical object itself, and, when introduced into the theory, has followed a course similar to other mathematical objects – theoretical structure and use in other branches of Mathematics – and nowadays its status is that of a regular mathematical structure.

5.1. Description of a table

Briefly, a table consists of headers and body data, located in rows, columns, cells. A generic table model (Figure 29) considers: title, top header, associated with the vertical region (column); left side heading, associated with the horizontal region (rows); an upper left corner which can eventually be empty. From the content, a generic model considers, in the first left column, rows of categories of the variable (optionally with header title type in the top cell of the first column, and, from the second column, the data values), and a body of data (which does not consider the first row nor the first column).

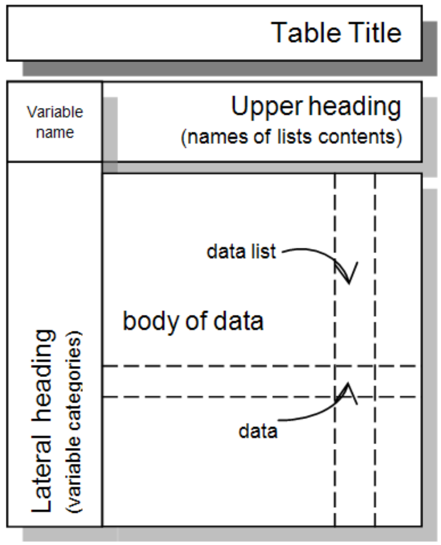


Figure 29. Generic table.

Whereas the table display is a logical relationship that presents the data, it is also possible to distinguish cells, rows, columns and blocks (of cells, rows or columns) (Figure 30).

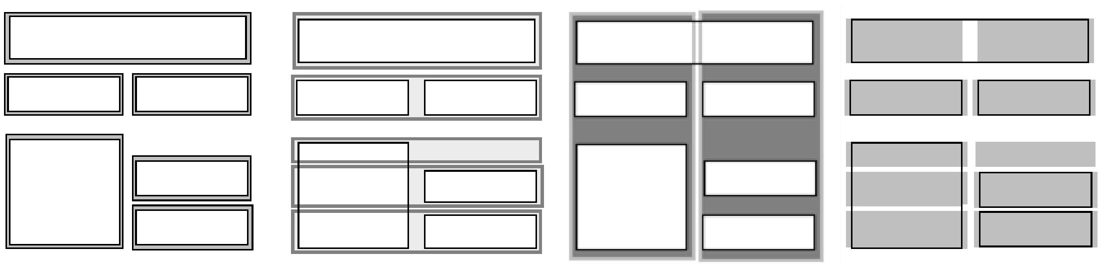


Figure 30. Logical elements of a table.

To comprehend a table in this sense requires consider that a relation *R* from *A* to *B* is a function $f : A \times B \to V, F$, (where *V* means true, and *F*, false): if *aRb*, then *f*(*a*,*b*)= *V*, otherwise, *f*(*a*,*b*)=*F*.

One of the ways to devise a mathematical object table as shown in Figure 31:

	b_1	b_2	b_3
a_1	d_{11}	d_{12}	d_{13}
a_2	d_{21}	d_{22}	d_{23}

Figure 31 Table example.

Let, for example, $A = \{a_1, a_2\}, B = \{b_1, b_2, b_3\}, C = \{d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}\}$. The table may then be seen as a function $f : A \times B \to D;\ f(a_i, b_j) = d_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 3.$

If we make $D_1 = \{d_{11}, d_{12}, d_{13}\}, D_2 = \{d_{21}, d_{22}, d_{23}\}$, the table can also be regarded as $F=\{f_1,f_2\}$, where

$f_i : B \to D;\ f_i(b_j) \in D_i,\ 1 \leq i \leq 2,\ j \leq i \leq 3$.

You can replace the side headers for the upper ones, provided adding a similar condition on the columns (instead of rows). Changing rows by columns can be considered an involutive operator on the table: twice in succession gives the identity.

Rows and columns can be associated with the concept of *tuple*, an ordered list of items; in addition, since a table is constituted by at least one list of data associated with a category of the variable, then each tuple is the data set per individual for that category.

One can also consider a collection of functions f_{ij} defined on $A \times B$, made of atomic pieces of the previous function *f*, which can be reconstituted from these.

Since Maier (1983) and Tijerino et al. (2005), the choice alternatives mentioned were slightly modified, and a *canonical* table is defined as follows: There is a finite set of headers $L = \{L_1, ..., L_m\}$; to each L_j, $1 \leq j \leq m$, is associated a set D_i, the *domain* of L_j[9]. If we make $D = \bigcup_{i=1}^{n} D_i$., then a *canonical table* is a set $T = \{t_i : L \to D \,/\, i = 1, ..., n\}$, such that $t_i(L_j) \in D_i,\ 1 \leq i \leq n, 1 \leq j \leq m$: each header is associated to a content, with the condition expressed.

[9] Nótese el cambio en la terminología.

5.2. Operations with tables

You can perform various operations with tables, for example:

$$T_1 \cup T_2, T_1 \cap T_2, T_1 - T_2, T_1 \Delta T_2 = \ T_1 \cup T_2 \ - \ T_1 \cap T_2 \ , T_1 \times T_2.$$

One example of $T_1 \cup T_2$ is the “pair of ecological tables” in Statistics, in which two tables are joined: the environmental variables table (in columns) and the data of the species table (in columns). Their side headers are identical and correspond to areas where the variables have been measured and the data (Cf. Thioulouse, 2011).

Very relevant is the *concatenation* of tables (Figure 32). A table T can be considered as the result of concatenation $T_1|||T_2$ of two subtables T_1,T_2 having the same lateral header, but whose superior header was separated into two and, correspondingly, the data body.

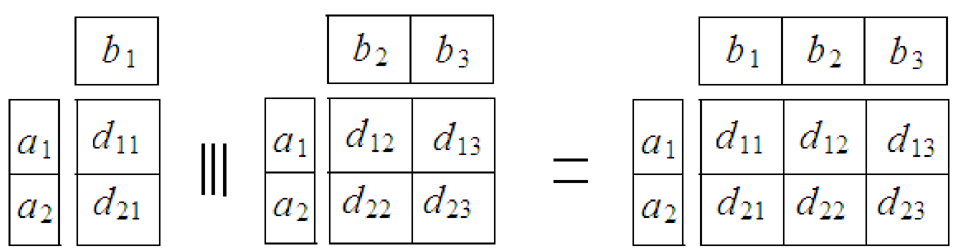


Figure 32. Concatenation of tables.

For environmental analysis it is necessary that environmental and species variables tables are concatenated associatively.

On the other hand, it can be considered the multiplication of (all) the elements of the body of the table by a number.

5.3. Algebraic structure

Recently, based on the above mathematical considerations and other of computer character, the theory of Table-algebra, an important branch of algebra, is being developed.

The natural environment to develop this theory is that of vector spaces, and it is convenient to take the field **C** of complex numbers as the scalars.

5.3.1 Associative Algebra

Recall that if A is a vector space of dimension n over C, of (ordered) basis $B=\{b_1,b_2,\ldots,b_n\}$, in order to build an structure associative algebra over **C**, it is sufficient to define the products $b_ib_j, 1\le i,j\le n,$ and extend by linearity and distributivity. In effect, if $v,w\in A,$ then

$$v=\sum_{i=1}^{n}\alpha_i b_i,\ w=\sum_{j=1}^{n}\beta_j b_j$$

so that we would have $vw=\left(\sum_{i=1}^{n}\alpha_i b_i\right)\left(\sum_{j=1}^{n}\beta_j b_j\right)=\sum_{i,j}\alpha_i\beta_j\ \ b_ib_j$

If $b_ib_j=\sum_{k=1}^{n}\lambda_{ijk}b_k,\ 1\le i,j\le n$, then the scalars $\lambda_{ijk},(1\le i,j,k\le n),$ are called the structure constants of *A*. *B* is still the base of algebra and | *B* | its dimension. The identity of A, if any, is written 1_A.

5.3.2 Table-Algebra

Let *A* be an associative algebra as above, $B=\ b_1=1_A,b_2,\ldots,b_n\ $, *A* commutative and with structure constants $\lambda_{ijk},(1\le i,j,k\le n)$.

A has an *table-algebra* structure if it holds that:

i. $\lambda_{ijk}\ge 0,\ 1\le i,j\le n$;

ii. there is an algebra automorphism *of *A* such that $v^{*\,*}=v; B^*=B$;

iii. to each $i,j,\ 1\le i,j\le n$, $\lambda_{ij1}=0$ if $b_i\ne(b_j)^*$ and $\lambda_{ij1}=\lambda_{ji1}>0$ if $b_i=(b_j)^*$

5.3.3 Applications

Furusawa and Kahl (2004) present the algebraic table-structure algebra as a basis for the mathematical interpretation of specific tables of computer use, and to obtain specifications for tables in a way that allows for proper implementation as data structure.

Moreover, table algebras are being used in other areas of Mathematics, a priori far from Informatics: they are used in graph theory (Arad et al, 2011); for the case of a finite group *G*, table-algebras are constructed about **C***G* algebras (Arad and Blau, 1991); the Sylow theorems of finite groups have been generalized to table-algebras (Blau and Zieschang, 2004). Additionally, and as it would be expected, the table-algebras themselves have been generalized (Cf. Arad, Fisman and Muzychuk, 1999), etc.

6. PEIRCE: TABLES AND THOUGHT

6.1 Logic tables

The truth table is a method used to determine the truth conditions of a sentence, that is, its meaning, depending on the truth conditions of its (mutually independent) atomic elements. In these tables the letters *V* or, correspondingly, the numeral 1 (true value), and *F* or numeral 0 (false value), appear. The truth table allows to determine in which situations the statement is true and what is false.

Charles Peirce conducted a study of the conditions of truth of propositions, work that continued throughout his career as a logician. In a manuscript[10] of 1893, in the context of his study of the functional analysis of propositions and demonstrations and his continued efforts to define and understand the nature of logical inference, Peirce presents a truth table that shows in a matrix form the definition its connective: the illation. Thus, Peirce's 1893 table is considered the first known example of a truth table in the familiar form attributable to an identified author, and precedes not only the tables of Post, Wittgenstein, and Łukasiewicz from 1920 to 1922, but also to Russell's 1912 table, and even to the tables previously identified by Peirce for the triadic logic of the period 1902-1909 (Anellis, 2012).

[10] "*An Outline Sketch of Synechistic Philosophy*", [Peirce *1893*; MS #946:00004].

Truth tables, together with Boolean algebra, opened mathematical developments such as propositional logic, binary logic, among others.

6.2 Abductive Thinking

Peirce (1903) sustains that *abduction* is the process of forming an explanatory hypothesis. Recognized as the only logical operation which introduces any new idea, since induction does nothing but determine a value, and deduction merely develops the necessary consequences of a pure hypothesis. He argues that the deduction proves that something has to be, induction shows that something is *currently* operating, and abduction merely suggests that something *may be*.

For Peirce the only justification of abductive reasoning is that from its suggestion deduction can draw a prediction that can be checked by induction, and, if we can learn something or to understand phenomena at all, this has to be achieved by abduction.
Considering the thought put into play by the tables, we take the idea considered by Peirce on reasoning, as there is a time of the flow of ideas when they appear one or several discoveries that enlighten; he said that "abduction is the first step of scientific reasoning " (Peirce, 7.218). If giving freedom to the mind we consider observing the data for a phenomenon to find and explain their behavior, abductive thinking is operating with, because in the logical operation novel hypothesis are emerging; a thought that uses the "reasoning for the best explanation" is activated. Abduction is the "first stage" of interpretation, followed by deduction and induction as it looks plausible assumptions "forming explanatory hypotheses" (CP 5.171, 1903).

7. CONCLUSIONS

The tables were a useful tool to record evidence, to order, to generate information from them. They have been a means to capture and promote the creation of knowledge, to build tools to formulate, transmit and use it expeditiously. Tables such as the Almagest's and logarithms clearly display their status of contribution to the development of the theory. Today, the tables themselves are a mathematical object of independent development, and affect other areas of the discipline, its rapid development as a mathematical object now shows once again the importance of tables for Mathematics – now acting from the inside of discipline.

From this linear epistemic path of tables, we can observe them as *proto-mathematical objects* (cf. Table 1, e. g.), then as a useful tool for centuries to study other mathematical objects, thus *para-mathematical objects*, and that only recently have taken the status of *mathematical objects*, one study in themselves.

We think that a different way to appraise the importance of the use of tables is to consider the case of the *absence* of tables. Such an exercise is hypothetical in general, but there is, however, at least one example that could be invoked: the Megarian-Stoic school defined, in IV century B. C., negation, conjunction and implication in the same terms as they are used today, but without the use of *truth tables*, defined, as we saw, by C. S. Peirce in 1893. It seems clear that the use of these tables could have well simplified the discussion (and possibly could have prevented the misunderstanding of that school by the historians of the nineteenth century (Cf. Bocheński, 1961, e. g.).

The table as a signs system integrates graphic forms with the economy of space and text, and allows various readings and flows of them. In turn, this allows to establish the various relationships between objects treated, to find patterns and regularities, by which it activates by the ordering of the knowledge we have of the classification schemes and symbolic systems. This implies a significant change in modes of thought, now more complex. Such complexity affects the cognitive processes involved in its use, which in turn suggests the difficulty of learning the tables.

The transit from lists tables took millennia, which, according to Piaget's genetic epistemological view, could give an idea of the difficulty of the implicit cognitive process. On the other hand, from an social interactionist perspective we might consider that the specific knowledge "table" comes to be recognized – with periods of acceptance and resistance – as a common knowledge, since the interaction and communication schools scribes of Mesopotamia to the scientific academic communities of today.

7.1 Some reflections

The epistemological reflection on the changes of meaning/functionality of the table concept can show us some features of the dynamics of the evolution of thought and knowledge, as we briefly review below.
This study allowed a glimpse of the epistemological evolution of the table in its dual role, of tool and of object. As a tool, the table presents distinctive characteristics throughout history, first and long-lasting as a repository of memory, in the tables of census data and in metrological tables. This role does is not manifested in the astronomical tables of the Almagest of Ptolemy and others, since not only periodic data were stored in them, but constituted more than a mass of data, and were a physical media to view and search for regularities and explore abnormalities of data in the phenomenon investigated, and were tables used to analyze the data.

A new perspective of simultaneous but distinctive features of a table use is given by the *quipu*. This accounting system through ropes and knots in cultures without writing seems to contain some of the features of associating an amount (knot) to a quality (color or direction) in an order (column or row), and allowed to search and to control data to obtain information. Besides the *quipu*, and as shown in the right image of Figure 20, it was also used, simultaneously, the yupana (tool to calculate). This means that the person occupying the role of accountant and treasurer, in the words of the chronicler, occupies an accounting record (with tabular sense, say) in conjunction with a device in a table format that was used to do arithmetic. The figure cited, therefore, provides us a picture of the origins of Andean culture, a little over 400 years ago, in which the responsible for recording data in memory and to make accounts, at least uses separate types of tables in two of their roles as tools, as a repository (*quipu*) and as computing device (*yupana*).

In addition to the features, primarily of memory repository and secondarily for analysis, who held the table since its beginning, it gradually began to be employed as a calculation tool to operate quickly with the values entered in it; the table became the forerunner of calculating machines, an issue that contributed to and accelerated the development of theories. An example is given at the beginning of the Descriptive Statistics: through a fruitful dialectics between tables as data repositories and tables for analysis, John Graunt (1662) uses the tables of census records and establishes classifications of causes of death and makes the first tablets about life chances. On these tables worked notorious groups of scientists who settled some of the initial basis of probability, and were among the pioneers who settled a systematic and different way of working and thinking phenomena.

Peirce's truth tables are tables to analyze and to reason, such as the contingency tables occupied by Pearson in 1904, which allow to infer about the distribution and association of two variables. The process of thinking and reasoning is based on inferences seeking to establish regularities, habits and beliefs (Wirth, 1998).

To this day, tables flaunt their roles of repository, calculation and analysis; they are a means for reasoning. New technologies allow the efficient construction of data tables customizable in content, operations and style, that invite to recognize and explain patterns of behavior; they are tables built to simultaneously record, calculate and analyze, and allow start to solve a problem by abductive reasoning from dynamic scans.

CHAPTER IV

Towards a Didactic of Tables

Yahweh said to Moses, "Come up to me into the mount stay there,
and I give you the tables of stone - the law and the commandments
I have written for their instruction"
Exodus 24:12

Chapter Summary

This chapter seeks to outline the status of tables in schools. It applies an epistemological lens to tables as a significant element in the analysis of knowledge flow and normalization of understanding. Combining table features from statistics and computer science, we propose a generic model for tables. We show once again aspects of tables as a calculation tool and a heuristic tool to explore new situations, and we investigate cognitive aspects, by studying tables as representations that support the construction of meaning from data, identifying individuals' roles and cognitive processes associated with statistical tables**.**

In order to approach "a didactic" of tables, we study their role in international testing items such as those of the TIMSS, and their status in the Data and Probability topic of the Primary Education Mathematics Curriculum, (MINEDUC, 2012), and in three OECD countries[11].

INTRODUCTION

Whereas one of the everyday processes of users of computer networks -that is, a large percentage of the population- is the extraction of information, the first section considers some features of the tabular structure in Computer Science. Then, as a result of information gathered during the previous chapter, a generic model of the table is set and a concatenation of tables considered as an object of study through an elementary school homework statistics is constructed.

The second section details what is meant by a statistical table from a semantic perspective and in technical language. After summarizing the historical background of the table format in statistics, statistical criteria for displaying tables are shown, and those which could be seen as more relevant to teaching are indicated.

The third section deals with aspects of teaching and learning tables since they emerged in the early schools of the scribes as a tool for solving problems in the schools of Nippur and also as a tool that normalized knowledge among the scribal schools of that region. We finish this

[11] Curricula of countries that are successful in international tests, and explicitly include the table object in their curricula.

section by examining the status of the table in the current Chilean mathematical curriculum related to statistics, and later in the curricula of England, Brazil and Singapore.

The fourth and final section analyzes tables in statistics items released from an international mathematics test and explores the cognitive demands of various tasks with tables according to student performance levels. Some taxonomies of statistical understanding are explored, and the chapter ends with a specification and description of the role of subjects faced with tasks involving tables.

1. Tables In Informatics

1.1. Introduction

The epistemological study of tables in the previous chapter showed us the peak of tables' use as a handmade computational tool, boom times that declined with the advent of mechanical calculating machines. One of the milestones in this transition is Babbage's complaint about the errors in manually constructed tables, leading to the beginning of table automation. Babbage's first project was the difference engine, whose main objective was to calculate logarithm tables; this tabulating machine was completed by Scheutz (1785-1873). Then Hollerith tabulating machines appeared, which are the origin of the current IBM.

Nowadays we deal tables everywhere. Visible or invisible, they are a basic structure of informatics. Current tools include graphic elements in the tables themselves that provide information and lead to comparisons along rows or columns. For example, coloring in different color scales, using bar lengths to highlight distribution and variation presented by numerical data, using sets of icons to classify according to categories, using a circle proportional in area to a numerical value within each row, angling the header names, or including rotated histograms.

This section provides some conceptual clarifications of tables in this area, which allow us to understand the complexity of the table structure.

1.2. Table Structures: Physical, Functional, Semantic, and Logical

An overview of tables in computer science allows us to consider them as the main objects of the databases that are used to store, organize and display data. Tables are composed of two structures: records and fields. A <u>record</u> is each of the rows in the table, and each record contains data of the same types as the other records. A <u>field</u> is each of the columns that make up the table and contains data of a different type than other fields.

This study allows us to understand the table as a tool and an object, in the sense of Douady (1986). A table is visually recognizable by its segmented rectangular grid, the cells of which contain headers and/or values (associated with one or more variables). Similarly, the table is recognizable through its relational structure, and thus it is possible to understand that "a table is a visual manifestation of a logical relationship" (Green & Krishnamoorthy, 1996 in Hurst, 2000, p.38).

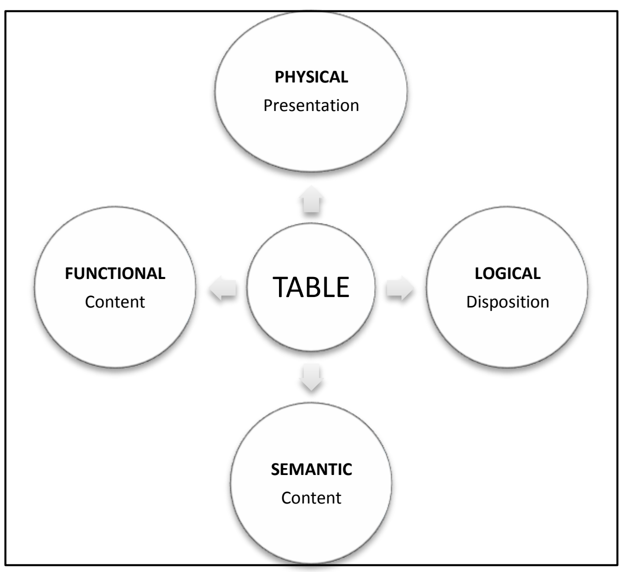


Figure 33 Aspects of the table structure

In Figure 33 it is possible to differentiate various aspects of the table as a concept: on one hand, a physical structure that makes it recognizable; on the other hand, its semantic structure (its content and meaning); additionally, its logical structure (invisible and related to the localization coordinates and type of content); and finally, its functional structure (the purpose of creating of the table). The general ideas presented have been obtained from literature in the area of computing (Hurst, 2000; Zannibi, Blostein & Cordy, 2003; Embley, Hurst, Lopresti & Nagi, 2006).

Figure 34 has been designed to identify distinguishing elements of a table regarding its contents and their location within the table.

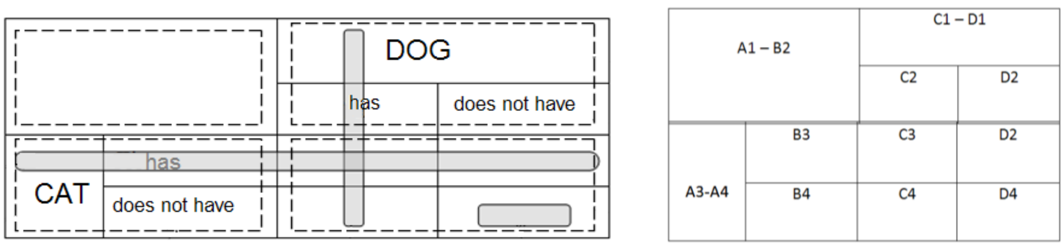


Figure 34 Differentiation of some components of a table (double entry table).

Physical structure of a table

This structure is formed in terms of the physical relationship between its basic elements: a table as a rectangular grid of rows and columns, whose intersections correspond to cells. The physical structure includes line drawings of the network, areas and angles that visually make up what is recognized as a table.

Logical Structure

This structure refers to the organization of the cells as an indicator of the relationships among them, the author's intention, and the restriction of two-dimensionality. The logical structure considers the syntax of the table, the arrangement of cells, rows, and columns, merging and/or splitting of regions, sorting, and indexing (see Figure 35).

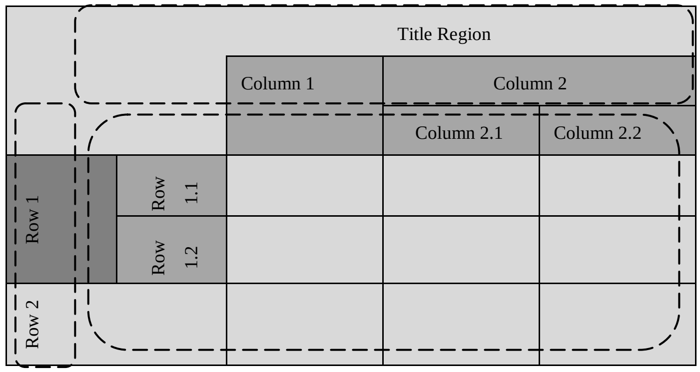


Figure 35. Logical Structure of a Table

Functional Structure

This structure is focused on the purpose of the person reading the table. Therefore, it focuses on access to table identifiers to access cells (row and column headers), access to data, and the identifiers of the data cells (body of data).

Semantic structure

This structure responds to the meaning of the text in the cell, the text object in the cell, and the meaning of reading the table. Spatially, it considers headers relative to the data area of the table, variable categories and subcategories, and inter-cell relationships (see Figure 36).

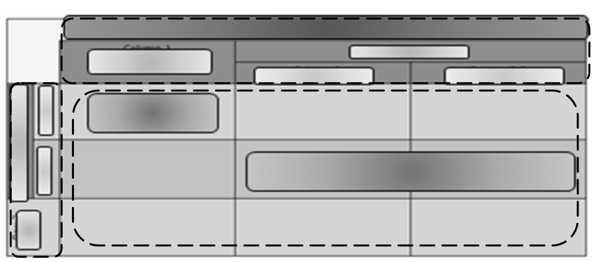


Figure 36 Semantic structure of a table.

1.3. The tables on the Web

In web pages, tables are used to organize and to improve the format of text and graphics; they can be created through a web developer or a computer programming language such as HTML.

Users are presented with large volumes of data intended to support decision-making processes in many areas such as e-commerce, database analysis, and search processes. The information displayed must manifest itself as important, draw attention, and facilitate the identification and integration of data (Resnick & Fares, 2004).

Regarding tables, [interface] designers have also focused on visualization techniques to facilitate online tabular presentation of products. Research by Fares and Resnick (2004) found that for focused or integrative data analysis, it is more beneficial to use a color-coded system instead of a ranking system, and that the use of both techniques overloads to some degree the information provided by the table.

Other research efforts related to tables are found in the study of human-computer interaction. For example, Hur, Kim, Samak, and Yi (2013) compared three ordering techniques (column order, simultaneous column order, and ordering by all columns with a vertical location) utilized cognitively by humans faced with a table representation to choose objects with multiple attributes. Using eye-tracking, the strengths and weaknesses of the three techniques were studied. Among the findings, they recognized that people suffer from an occlusion problem in sorting by all columns with faithful vertical location for some low-level analytic tasks.

1.4. Generic Table Model

Lists are the basic units a table. They comprise enumeration and/or classification and consider a column disposition (vertical reading) or row disposition (horizontal reading). They have no header, and their components are separated by spaces and/or punctuation.

This study includes a table that has a rectangular shape physically composed of rows, columns and cells, in which you can distinguish a top margin (first row) and lateral margin (first column) that are completed with headers, and a central area complete with the body of data. As already outlined in section 3.1 of the previous chapter, the data for each row is a data class, and headers make class names - or variable categories - explicit using a specific written, graphic, or symbolic label in the side margin. Tables sometimes include notes to help understand the data, for example the meaning of an icon used in a table. We propose a generic table model (Figure 37) that considers the title, the lateral header area associated with the variable categories, the superior header, an upper left corner that eventually identifies the variable, and a physical network of columns and rows that generate cells containing the body of data.

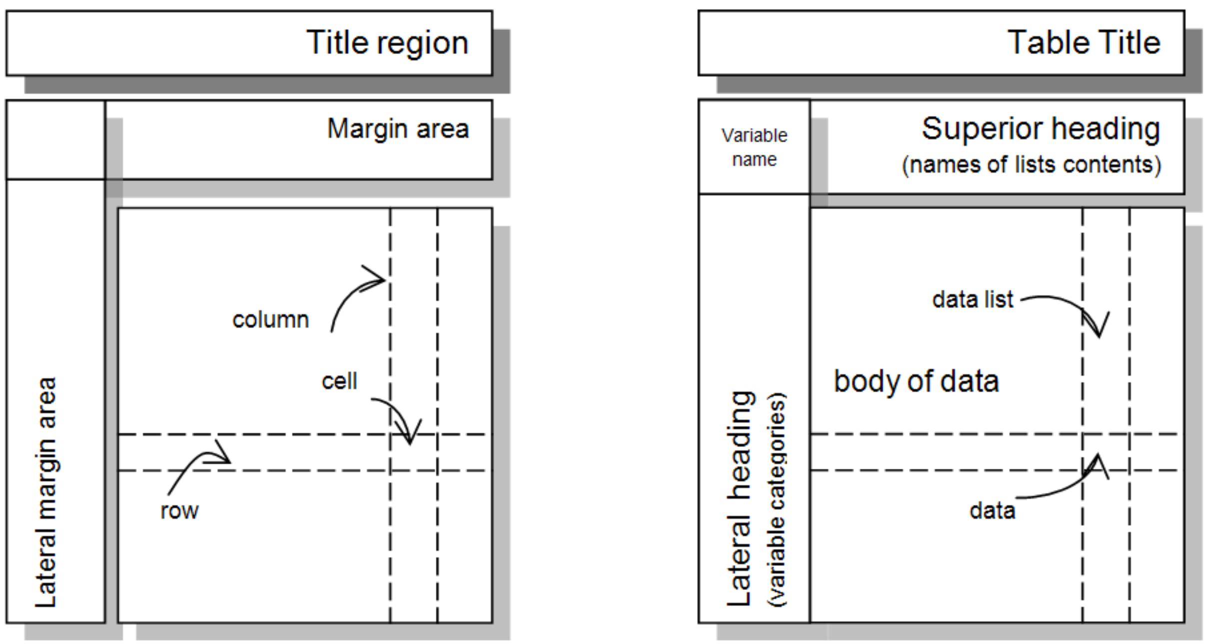


Figure 37 . Generic models of table as physical and content structure.

A table is constituted by at least one data list (Duval, 2003) associated with a variable category. Tables, as a device organizing partitions and the resulting classes, show varied aspects. For example (see Figure 37), a statistical table (e.g., the frequency distribution table) comprises a network of rows and columns used to present data in an organized and summarized manner, corresponding to one or more variables related to a phenomenon, allowing for displaying the behavior and comparing the data, thus providing specificity in the understanding of the information that can be extracted.

1.5. Concatenation of tables in a textbook

In Chapter III some table operations are specified, including concatenation. In a computer-science article, Furusawa and Kahl (2004) exhibit an algebraic structure, called an algebra table, as a basis for the interpretation of mathematical tables. This article begins by introducing the concept of a compositional table. Then it provides a reference to the specific algebraic notation used and shows how this can be applied to basic tables. Then it discusses nested headers, motivating the general definitions of algebra tables, and finally it uses the free algebra machinery for table specifications in a manner that allows for proper implementation and data structure.

The purpose of this section is to show the concept of table concatenation using a school-level table from the "Tables and Graphs" section of a grade 3 mathematics textbook[12] to develop *composition* from its components.

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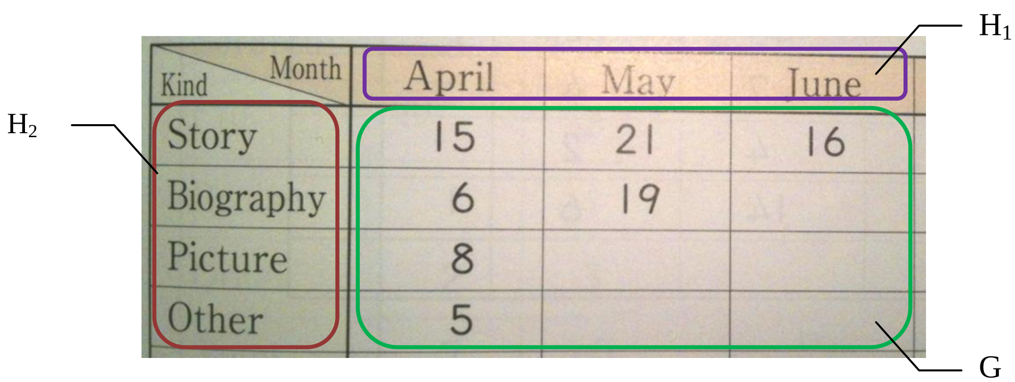


Kind \ Month	April	May	June
Story	15	21	16
Biography	6	19	
Picture	8		
Other	5		

Figure 38 T_f table with headers and body of data, resulting from concatenation.

Let the table be T_f, its headers H_1 and H_2, and its network G. T_f is the result of concatenating two subtables, T_{fa} and T_{fb}, consisting of its headers H_{1a} and H_{2b} and networks Ga and G_b (see Figures 38 and 39).

[12] Book "Study with Your Friends, Elementary School MATHEMATICS", 3rd grade, vol.1, pages 65-66. Tokyo: Gakkoh Tosho Co., Ltd.

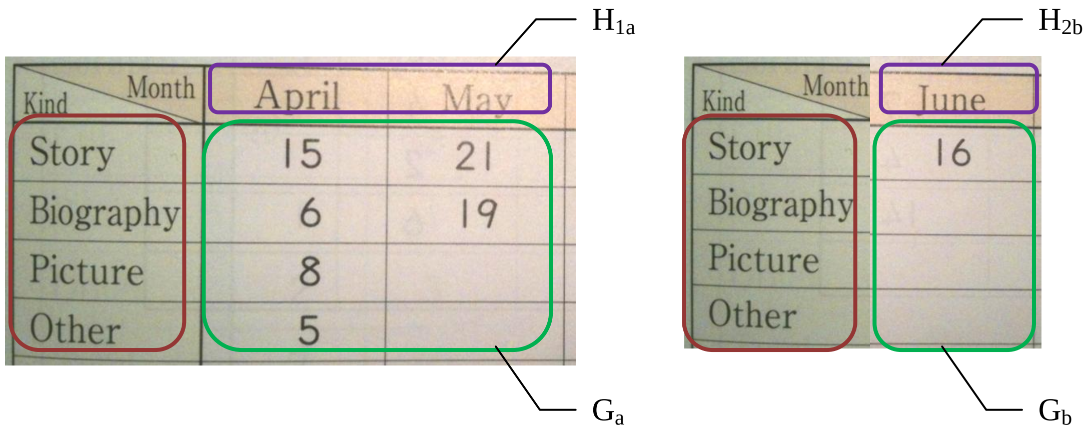


Figure 39 Process of concatenation of two sub-tables, $T_{f,a}$ and $T_{f,b}$.

The table T_{fa} can be regarded as the result of adding a single header to a table of dimension one (see Figure 40).

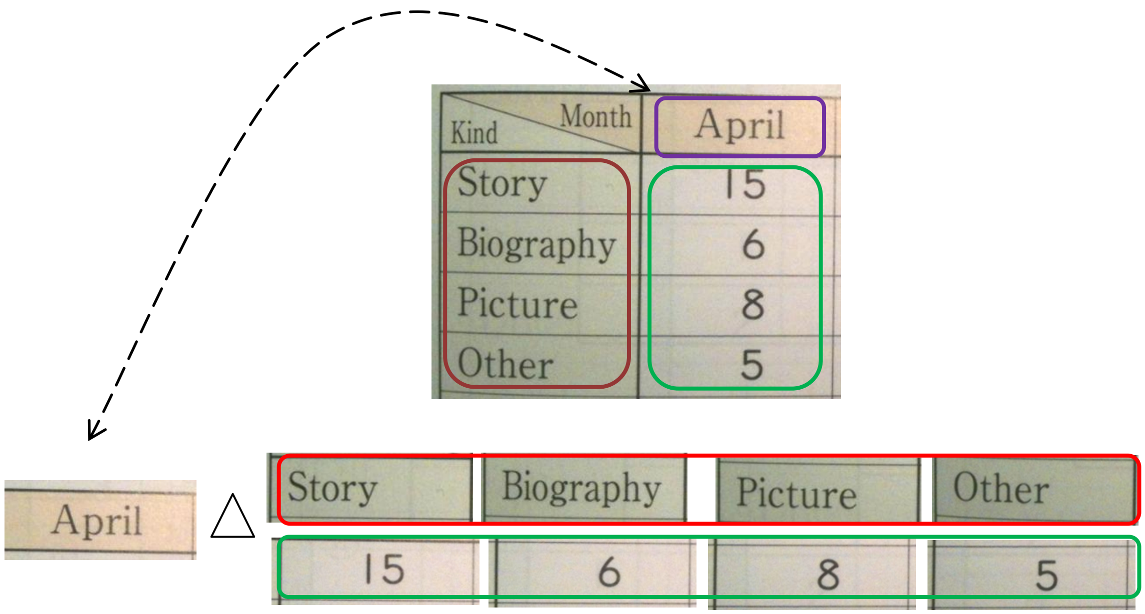


Figure 40 Concatenation process.

The ▷ operator adds a single header in new first dimension. The one-dimensional table can be seen as the result of successive horizontal linkages (see Figure 41)

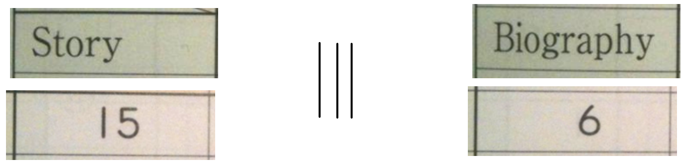


Figure 41 Successive concatenations processes.

The individual components are the result of adding a single header to a zero-dimensional table, or cell. Thus, schematically:

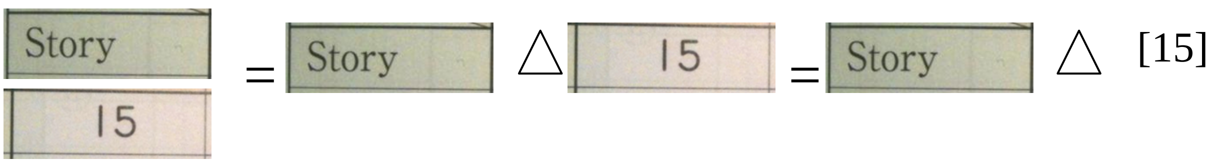
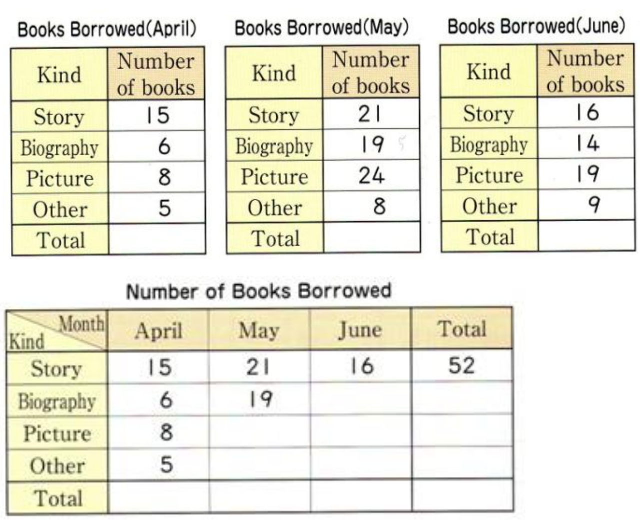


Figure 42 Tabular expression as a result of the concatenation.

Using the | | | operator for horizontal concatenation, and the ▷ operator to add headers and [ ] to build cells, the T_f table can be written as a tabular expression (see Figure 42).

Figure 43 shows the task in a grade 3 Japanese textbook, which shows the concatenation of tables described.



Books Borrowed(April)

Kind	Number of books
Story	15
Biography	6
Picture	8
Other	5
Total	

Books Borrowed(May)

Kind	Number of books
Story	21
Biography	19
Picture	24
Other	8
Total	

Books Borrowed(June)

Kind	Number of books
Story	16
Biography	14
Picture	19
Other	9
Total	

Number of Books Borrowed

Kind \ Month	April	May	June	Total
Story	15	21	16	52
Biography	6	19		
Picture	8			
Other	5			
Total				

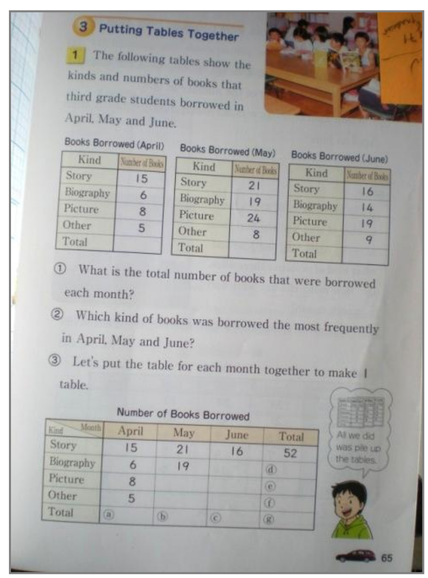


Figure 43. "Joining Tables" from "Study with Your Friends, Elementary School MATHEMATICS" workbook, 3rd grade, vol.1.pages 65 and 66. Tokyo: Gakkoh Tosho Co., Ltd

This task is a vehicle for exemplifying the table as a mathematical object[13] because the table can be defined, it includes properties, and operations can be performed on it.

2. TABLES IN STATISTICS

2.1. Introduction

The inclusion of statistics in the school curriculum from the earliest grades shows the importance given to it in scientific development. The construction and understanding of tables are part of statistical literacy. Some studies argue that although statistical tables are important tools for communicating information, they receive little attention in research, education, and statistics practice. This lack of appreciation could undermine the quality of tables that accompany scientific and non-scientific presentations, which can affect statistics learning and teaching (Koschat, 2005; Schield, 2001; Wainer, 1992; Feinberg & Wainer, 2011; Ehrenberg, 1977, 1978, 1986 and 1998).

Today's society demands statistical literacy of its citizens, and the ability to read and understand tables in particular (Friel, Curcio & Bright, 2001). Tables as well as the graphs are prominent in scientific production, such as in the processes of modeling and specifically in exploratory data analysis (Tukey, 1977). Data analysis activities are related to fundamental mathematical ideas and processes of counting, measurement, and classification (Russell, 1991), processes for which tables are common tools. Beyond being present in science, tables are part of everyday life, are found in the media, and are used as common IT tools, and in technological, commercial, and agricultural tasks to mention some of the main areas of the economy. The ability to interpret information presented in tables is a key element of scientific culture, and this requires explicit attention as part of the education of all people. Some people cannot work correctly with quantitative information and can become dependent on the interpretations of others, becoming vulnerable to intentional or unintentional deception (Dewdney, 1993), since they have not realized that data can be manipulated to support certain views.

[13] Chevallard (1985) noted that as knowledge objects mathematical notions provide: a definition, properties, usage occasions, etc.

Moreover, information in which we are not looking for trends but rather for the actual data ("hard data"), such as the delivery of voting information in elections or the day's sales data, are examples in which the characteristics of tables outperform graphs. However, this difference is not important and is reflected in educational discourse. As Koschat (2005:31) notes, the benefits of graphing information are widely known and the characteristics and principles of how to make a good graph are taught. However, the principles of how to make a table are ignored.

2.2. A brief history of tables in Statistics

As noted in Chapter III, the first data table in history that has the above structure comes from the Sumerian city of Shuruppag, dating back to 2600 BC, and has ten rows with three columns. The first two columns correspond to lists of measurements of length and the last column provides products to measure area (Campbell, Kelly, et. al, 2003).

Discounting census tables, in the area of statistics records in empirical data tables are recognized in the early 1600s, a time when the publication of number tables appears. In Germany, "Die Tabellen - Statistik" emerges as a branch of statistics devoted to numerical description of facts. In 1662 John Graunt used the mortality figures in London to create the first life table or mortality table. In 1671, Jan de Witt made the first attempt to scientifically determine the purchase price of annuities, using mortality tables. Two decades later, in 1693, Edmond Halley constructed the first real mortality tables, which contain the ages at death of a sample of individuals under stable conditions. In 1779, Lambert designed the first semi-graphical visualizations combining tabular formats with graphs; Minard developed the table-graph in 1844 to show commercial traffic; and in 1883 Loua followed with the first known use of a semi-graphical table showing a data table with levels of shading. Earlier, in 1835, Quetelet used a variety of frequency tables of population data upon which he based his concept of "the average man", as well as crosstabs, although without considering marginal characteristics. Bortkiewicz did something similar in 1898, using contingency tables[14] for the

[14]Contingency tables, a term coined in 1904 by Pearson, are crosstabs, or 2x2, whose main objective is to analyze the relationship of two or more qualitative variables, and typically include the marginal totals of both variables and the concept of variable dimensional statistics.

deaths of soldiers kicked by horses in the Prussian cavalry and arguing for the Law of Small Numbers (Poisson).

In 1904, Pearson developed statistical theory with contingency tables for bivariate frequency distributions and measurements of the association of two variables. In this period it became evident that statistics as a discipline would continue to build and develop distribution tables.

The Tukey Data Analysis (1977) considered the tabular tools used to discover patterns in the behavior of the data supporting the establishment of hypotheses. Tables, or combinations of tables such as "sparklines", a combination that shows graphic information in a table in-line with content, which Tufte designed in 2004, would continue to be developed and used in this way up to the present (Friendly, 2009, Stigler, 2002; Pearson, 1904; Quetelet, 1835).

2.3. The notion of a statistical table

A review of the previous section, allows us to recognize tables as a polysemic object in terms of the plurality of word meanings and to value them based on the diversity and frequency of their use in statistics. In Exploratory Data Analysis, EDA (Tukey, 1977), or modeling, tables and graphs are prominent at the beginning and end of the study. This is because in the development of statistical analysis, initially the data, data sources, and unusual features are explored by displaying data in tabular and/or graphical formats. Then, after carrying out further analysis and completing the study, the results of the analysis must be communicated to the target audience completely and concisely.

The task of developing and interpreting tables is an integral part of scientific practice in academic articles and reports where the results of statistical analysis are reported and tables and/or graphs are often included. The construction of tables that can be read easily (at a glance) not only helps novices in reading tables, but also to experts.

The tables differ in variety, structure, flexibility, notation, representation and use, characteristics that let them cover a wide range of functions, and make them a widely used format. For example, many statistical reports and research papers exhibit more space devoted to tables than to graphs (Feinberg & Wainer, 2011).

Tables, as a format for displaying information and/or as a transition tool to plot data, receive little attention as a topic of research and education. Several statistics researchers have studied graph understanding as a research area, but do not study tables. Few researchers have addressed the issue of tables in statistics, however the contributions of Ehrenberg (1977, 1978, 1986, 1998) and Wainer (1992 and 2011) must be recognized, as well as Tufte and Graves-Morris (1983), Tufte (1990, 2006), Schield (2001) and Koschat (2005).

As stated, we understand a statistical table (e.g., a frequency distribution table) as a rectangular array with a structure comprising a set of rows and columns, which allows data representing one or more variables (characteristics of the phenomenon being studied) to be presented in an ordered and summarized manner to allow for the visualization of the data's behavior and facilitate the understanding of the information that can be extracted.

Some authors classify statistical tables based on the number of variables they represent, namely, one-dimensional or single entry (one variable), known as lists (vertical or horizontal), two-dimensional or double entry (two variables), and multidimensional (three or more variables), see Figure 44.

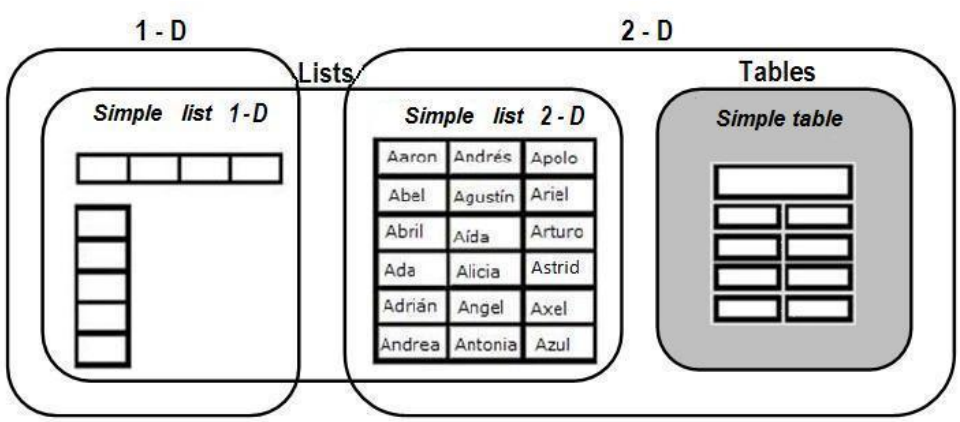


Figure 44. List and Tables according to dimensionality.

The structure of a statistical table

Since the purpose of a table is to communicate, it must necessarily have a title that summarizes the main idea. This title should be complete, clear, and concise, providing the context of when and where the study was conducted, and, if applicable, the sample size.

The *body of data*, defined in an inner rectangular block consisting of a group of cells formed by the intersection of rows and columns, usually contains numeric information, can be located by the subscripts of rows and columns. As noted, the top row and the far left column are usually not part of the body of data.

The *lateral header* or *first column* reflects different variable categories according to its classification. If the table represents more than one variable, the lateral header or the first column generally represents the variable with more classes or categories, or, in causal studies, the variable that is the determining factor.

A table, in its simplest version, is a structure in which numbers and text in rows and columns are arranged, often with a row corresponding to a case and a column corresponding to a variable. For a single-variable table, whether qualitative or quantitative, the name of the variable is located in the header of the first column. Corresponding categories are located under the variable name. If the variable is qualitative, the different categories it can include are placed here. If the variable is quantitative, discrete, and takes on only a few different values, the different values of the variable are placed below the name of the variable. In the case of a continuous quantitative variable and/or a quantitative variable with many different values, intervals are located under the variable name.

The *superior header* contains the name of the content of the columns, for example, frequency measurements, or other variable summaries. Totals are placed in the last row and/or the last column, sometimes called marginal totals (usually sums, averages, or percentages).

Another kind of statistical table is the 2 x 2 table, in which the values for two qualitative variables are crossed (initially these may be simple tables for each of the variables). For example, for the possession of a pet of type A and/or type B, the categories may be “has this kind of pet” or “does not have this kind of pet” (subcategories that must be exclusive and exhaustive). The frequencies can be placed in simple tables for each variable, one for each pet, or placed a single frequency table that "crosses" the variable categories, given that the intersection is not empty, as shown in Figure 45.

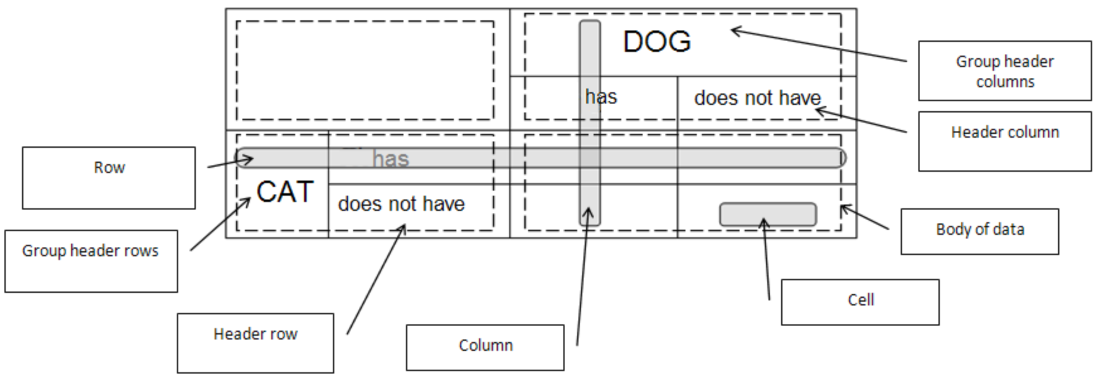


Figure 45 Structure of a 2x2 table

2.4. Some criteria for building statistical tables

In 1977, Ehrenberg distinguished data tables according to three types of purposes: informal work tables (for use by expert analyst and colleagues in the area, without considering a wider audience), tables for supporting or illustrating a specific conclusion or findings to a more-or-less specific audience, and tables created for recording data used in official statistics.

The basic rule in the construction of a table that it be visually easy to understand. When reading a table, short-term memory and processing information routines are used, and therefore, in reading the contents of numerical data in a table it is useful to focus on the variation of a single row or a single column, preferably those with summary measures such as averages or marginal totals (Ehrenberg, 1977, 1978, and 1986).

In 1998, Ehrenberg outlined five criteria for the preparation of tables in order to turn data into information and better communicate the table's purpose. The rules are based on short-term training to better address the memory task, as in reading a table of numbers you have to remember some of them, at least briefly, and do some mental arithmetic. The criteria proposed by Ehrenberg are:

1. Sort rows and columns according to a numerical order (this order lets us see whether they follow a pattern or not, and this approach helps make it easier to see, remember and relate).

2. Round to two digits (numbers with few digits are easier to read and remember, this approach facilitates mental math and retention values).

3. Deliver averages as visual focus (rows and/or columns with averages help provide a visual/mental focus; this approach allows for comparisons and examining relationships).

4. Use a tabular layout to guide the eye (single-spaced rows with some wider spaced horizontal lines, selective vertical and/or horizontal rows interspersed with shading, and some words in bold, this approach helps to view and retrieve information).

5. Deliver a written summary sentence (communicating in words what the numbers in a table deliver, helps the reader, for example, "All the numbers decrease." This approach allows the reader to better see and remember patterns, as well as to verify and detect irregularities).

The result of an analysis is influenced as much by the data as by the analytical assumptions and choices performed during the statistical analysis. It is beneficial to understanding the problem to complement the presentation of formal analysis with an informative tabular presentation of the data, either in its original form or as a numerical summary.

Numbers often require less explanation than the model's constructs. In many areas, analysts and users can easily understand the numbers presented to them in a context where quantitative data has an immediate meaning. There are good reasons to display numerical information in a simple structured format, both to support the communication of the results based on a model and to complement graphs.

Koschat (2005) specifies three considerations for building tables: the choice of columns and rows, the number display, and simple graphic elements. With respect to the rows and columns, he recommends that rows to be compared be close to each other, that numbers be limited to five or fewer digits, and that commas or spaces be added every three numbers. Considering that an entry is characterized not only by its value but also by its position in the table, he suggests making prudent use of lines and shading, or use of different fonts or spacing, or possibly shaded bands to help determine the position of items being compared.

Cook and Teo (2011) selected criteria referred to in Ehrenberg (1977) and Wainer (1992) as a way to assess how well some tables in three statistical journals were built. They considered: the number of entries (the fewer, the easier to understand), the maximum number of digits (not counting the zero before a decimal point), decimal alignment (preferably aligned), vertical versus horizontal comparisons (preferably vertical), and the presence of parentheses (which often provide unnecessary visual confusion). This research found that less experienced statisticians perform better in extracting information from graphs than in extracting information from tabular forms.

2.4.1. Other criteria for displaying tables

According to Wainer (1992) the basic steps to understand a table include extracting the basic units of information, observing trends and groupings, and comparing between groups. Meanwhile, in 2011, Feinberg and Wainer examined the presentation formats used in a scientific journal during the period 2005-2010 and found that the tabular format was dominant. After a critical analysis of the tables, they found that most of them might be more understandable if they had considered rounding, used statistical measures and had a sorting criterion.

Regarding considering rounding to facilitate understanding tables, subjects do not understand more than three figures with ease and more digits are rarely warranted to further statistical accuracy (Feinberg & Wainer, 2011). As for statistical measures, the authors argue that in most tables it is useful to include some, such as sums, means, or medians, and advise that the rows or columns (or both) that include them be separated from other entries, with bold face, spaces, or lines.

Tables must transparently show the results, and should be as autonomous as possible, in order to be submitted to the judgment of readers and scientific peers, for example, using the exact numbers with a minimum of significant figures, including the most important statistical measures of the results, arranging rows and columns to deliver information, using white space to suggest groups, clearly titling headers, an sorting by frequency value instead of an alphabetical order (Gelman, 2011).

2.4.2. Other considerations for presenting data

The literature advises us to display numerical information using sentences if you want to show at most five values, to use tables when displaying more numerical information, and to use graphs for complex relationships (Van Belle, 2011). A few numbers can be displayed through a list, while many numerical values should be displayed in a tabular format using summary statistical measures to show relationships in numerical data. However, these do not fully show the relationships, so it may be better to display the data using a graph.

Tufte and Graves-Morris (1983) suggest creating a table instead of a graph when there are many "localized comparisons", and believe that large tables can be an excellent means of communication. These authors suggest using phrases for relationships between two or three data entries, tables for more than three and less than 20 data entries, and graphs for three or more relationships and even more so the higher the number of data entries and relationships between them.

2.4.3. Relevant aspects for building tables

According to our literature review, the following ideas are related to the construction of tables.

Ordering: Since one of the most important processes carried out in tables is comparison, it is necessary to facilitate the spatial proximity of data given that comparisons between columns tend to be easier than between rows.

Grouping: Sorting by numerical attributes could show sets of natural groups in the data. Grouping by some predefined criterion allows easier comparisons between groups, or you may group by common sense, for example, time elapses from past to future.

Numbers: Limit the number of digits to show, especially in the case of some error measure - round considering that this involves a possible loss of information.

Measures: Submit statistical measures, such as the median, which is independent of the end values and appropriate for skewed distributions

Display: Consider that human visual perception considers position, shape, size, symbolism, and color, to eventually use shading, bold face, space or blank lines, lines, different fonts, icons, or coloring, whose use is valuable only if it helps to visualize the data's behavior.

2.4.4. Interpreting relevant aspects of tables

Considering that a graphical method is effective only if the decoding is effective (Cleveland & McGill, 1985), tabular interpretation is an abbreviation of the process of translating a visual representation into a verbal description of a situation that the table communicates, and comes from the communicative and primary intent that originated the construction of the table.

Kemp and Kissane (2010) propose five steps for interpreting tables and graphs. The framework developed by the authors has been used successfully in primary, secondary and tertiary mathematics education[15] and supports both students and their teachers and helps users to develop strategies to read these formats and critically interpret the information presented. The framework for table interpretation provides a progression from simple to more complex numerical reading interpretations, as detailed below.

Getting Started. Look at the title and read the headlines to know what is compared. Legends, footnotes, and the data source let you know the context and the quality of data expected.

[15] Framework used informally in Australia, Thailand, Singapore, Philippines, Czech Republic, Germany and Mexico

Taking into account the information on the questions raised in the study, the sample size, sampling procedures, and sampling error.

What do the numbers mean? Know what the numbers represent, find the largest and smallest values in one or more categories to begin developing a review of the information (percentages, etc.).

How do the numbers differ? Look at the differences in the values of the data in a single data set, in a row or a column, or a marginal row or column.

Where are the differences? What are the relationships connecting the variables in the table? Use information from the previous step to make comparisons between two or more categories or intervals.

Why do the numbers change? Why are there differences? Seek reasons for the relationships that have been found in the data, taking into account social, environmental, and economic factors; think about sudden or unexpected changes, and the local and global context.

This proposal provides a generic template for teachers to help their students develop strategies for interpreting data in tabular form, and can be applied to simple and complex tables. The level of complexity and the specific content of the table should be selected according to the types of data being faced by students as well as the concepts learned, so that conceptual understanding and meaningful interpretation of the information displayed by the table can occur.

2.4.5. Considerations

If tables allow for viewing data's behavior and help in creating graphs, the question arises: Are tables better than graphs?

"Less form, more content: that is what tables are about," argues Gelman (2011, p.6). This author believes that graphs can be distracting and could lead to error by showing convincing patterns that are not statistically significant. In the initial stage of data analysis, diagnostic graphs can be useful in developing a model, but final reports present tables. Graphs place the reader one step further from the numeric inferences that are the essence of rigorous scientific research (ibid.).

Koschat (2005) estimates that in tables there is less intervention by the analyst than in graphs or modeling. Some of the advantages of using tables to provide information are the presentation of data or a numerical summary of the data; the data in tables can be used and convert to other forms, such as a graph or model; and tables allow users to manipulate, operate on, and interpret the real numerical data.

Gelman, Pasarica, and Dodhia (2002) appreciate that data is presented in tabular form in a variety of contexts, and perhaps the main reason for using tables is that it is a known format. More than a third of the tables that appeared in a year in a statistics journal were summaries of frequency evaluations, which belongs to a specific task in statistical research.

2.5. Tables in the Statistical Education

To introduce statistical education, we recall that in the middle of the seventeenth century, John Graunt, the creator of life tables, was a pioneer in using fundamental statistical thinking to observe and try to explain the differences in numbers using context knowledge. From knowing the number of births, Graunt inferred the number of women of childbearing age and then estimated the total number of households and the average household size to produce an estimate of the population. Graunt discovered how to think and reason with data and therefore a new way of studying old and new demographic reports.

Pfannkuch and Wild (2004) argue that similar forms of statistical thinking arose independently in many parts of Western Europe in the same decade. Other pioneers, in the sense that they reasoned about their data, were: Petty, King, Halley, Hudde, Huyghens and Davenant, among others. This reasoning approach included estimating and predicting and then learning from the data, not only describing or gathering facts. This is a change in the use of the data tables from a recording tool to an analysis tool, on which policy decisions based on data are supported.

One of the objectives of statistics education is to provide a statistical literacy, which includes the use and interpretation of representations. Included among statistical representations we find tables and graphs. The tables that are used in school statistics are mainly frequency tables, distribution tables, and contingency tables.

The most widespread use of tables in schools and the curriculum is as a calculation and procedural tool, taught to be completed (with math operations) and searched (for fixed numerical answers). The current status of data tables causes the loss of their potential for reasoning and thinking statistically, as vehicle for observing behavioral patterns in the data's behavior and the application of tables' properties and conditions.

A line of research on the assessment of math teachers' errors in statistical literacy tasks, focusing on the detection of these errors in initial and continuing teacher training, addresses the resolution of reading and interpreting frequency distribution and contingency tables and graphs (Tauber, Cravero & Redondo, 2013). Estepa and Gea (2011) studied the errors and difficulties faced by students in the resolution of a covariance task. The authors report the incidence of the size of the crosstab: more stable results are obtained with 2x2 crosstabs than in studies of 3x3 tables. The study observed: failure to use all of the cells to form an association judgment in contingency tables; prevalence of the joint presence of the study variables in the association judgment over the joint absence of the study variables (cell to cell in contingency tables of dichotomous variables), presence of difficulties in understanding the non-uniform growth/decline in two variables shown in a two-way table. Meanwhile, Bruno and Espinel (2005) study how prospective teachers build a frequency histogram from a list of data. In their research they request that participants build a table of absolute and relative frequencies and a histogram from a data list. Only 11 of 39 future teachers represented the relative frequencies, although they had no difficulties in representing decimal numbers on the vertical axis. Batanero and Diaz (2007) show that most of the participating high-school students were able to solve problems identified in probabilistic format when taught to use double-entry tables to solve problems related to Bayes' theorem. Espinel (2007) and Espinel, Bruno, and Plasencia (2008) used a questionnaire referred to statistical reasoning involved in graphs and compared the results obtained for 137 future primary-school teachers in Spain with those obtained in a study of 345 American students; among their findings are that one question that resulted in a high level of difficulty was one that asked participants to choose from the graphs presented the one that best characterizes the shape of the distribution of the data shown in table form.

The emergence of statistical education as a research topic has allowed for the development of studies on the understanding of graphs and/or tables (Curcio, 1989; Friel, Curcio & Bright, 2001; Aoyama, 2007; Espinel, 2007; Tauber, 2006; Arteaga, Batanero, Canadas, & Contreras, 2011; Pfannkuch & Rubick, 2004; Chick, 2004), and studies based on characterizing the elements of statistical literacy (Gal, 2004; Ben- Zvi & Garfield, 2004; Schield, 2000, 2006) and statistical literacy in school (Watson, 2006; Burrill & Biehler, 2011) and in adults (Gal, 2002). Collaborative work between teachers in initial or continuing training in statistics education using technology in the classroom is studied by Ben- Zvi (2000, 2001, and 2007).

Other tables that are used in school statistics are the well-known stem-leaf diagrams and tally charts. In this study we consider stem-leaf diagrams and tally charts as tables because they contain all the individual data in a rectangular network, although with the consideration that the data they contain must be or has been transformed (in the first case the numeral has been broken up, and in the second, the tallies represent counting marks that let us visualize the order of magnitude of the associated number).

3. TABLES IN THE SCHOOL CURRICULUM

3.1. Introduction

The development of mathematics is not continuous. There are various social dynamics that gave rise to it, in different times and cultures. Some mathematical historians argue that the structures and systems of ideas included number patterns, logic, and spatial configuration (Joseph, 2011). According to the epistemological study outlined in Chapter III, the table has accompanied the development of mathematics and has evolved over time.

Mathematical ideas spread throughout the world through the centuries and in different cultures. There were cultural relations that led to the development of mathematics, although separate developments also occurred (see Figure 46). The following figure shows a general overview of some areas of major cultures and findings that have made contributions to delineating mathematics' early development.

Math was transmitted through writing on clay *tablets* or papyrus containing numerical developments in lists or tables. While tables were used in all these cultures, the Mesopotamians are recognized as the great table makers, something favored by the clay storage medium of their productions. As noted, tables were slow to establish themselves in the history of Mesopotamia. For a long period, scribes used mainly lists and, only rarely, tables. These appeared irregularly during the half millennium after the invention of writing, and then were reinvented several times, only really establishing themselves in the nineteenth century BC.

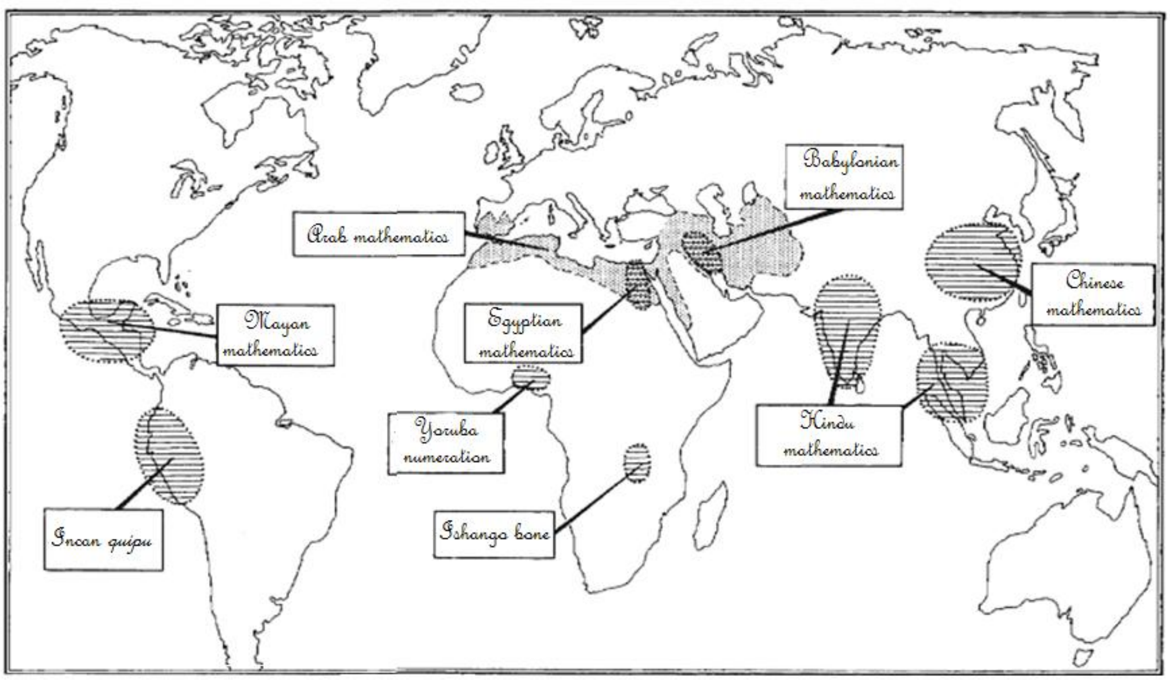


Figure 46. Some cultures which developed mathematics Joshep (2011).

The origins of the use of tables, and thus the transition from lists to tables, are located in the Old Babylonian period. Students in a scribal school have left particularly plentiful, varied, and complete documentation, which allow us to accurately reconstruct some stages of learning at that time. This documentation consists of thousands of drafts of school notebooks, i.e., written exercises on clay tablets. These school tablets, intended to be discarded or recycled were found in large quantities in the foundations of houses, walls, and floors. The most important documents found come from Nippur, the great religious and cultural capital of ancient Mesopotamia (Proust, 2009). As we said at the beginning, this chapter seeks to outline the status of the table at school; to contribute to this task, Chilean curriculum and curricula from some other countries will be explored.

3.2. School curriculum in Mesopotamia: school tablets in Nippur

Although school tablets have been discovered in almost all the major sites of the ancient Near East, most come from Nippur. This city has provided one of the few groups of school tablets that allow both qualitative and statistical analysis, and recent studies have enabled the reconstruction of Nippur’s "curriculum" (Proust, 2009). However, uniformity of content of tablets of school mathematics found in Mesopotamia and neighboring regions, gives the documentation of Nippur importance beyond the local level.

It has been found that education began with a first level, where young scribes had to memorize long lists, and then rebuild them in a certain order, in written form, and probably recite them orally. In the area of mathematics, these are

- metrology lists (enumeration with a progressive increase in measures of capacity, weight, area, and length - in that order);
- metrology tables (listing the same items as in the metrology lists, but including, before each item, its correspondence with a number written in place value notation);
- numerical tables (reciprocals, multiplication, squares, square roots, and cube roots)

After the first level, there followed a more advanced level dedicated to the calculation, that is, the algorithms for calculating multiplications, reciprocals, surfaces, and probably volumes. A rough idea of the proportion of numerical tables containing these texts can be taken from Table 5 (Proust, 2009, p.2).

Table 5

Distribution of the tablets found at Nippur as the texts contained in them.

Content of the tablet	Number of tablets
Metrological lists	187
Metrological tables	161
Numerical tables	417
Calculation exercises	38

The data in Table 5 show that the first stage of mathematics education focused on measurement notation, and metrology was an essential component, as the metrological tablets accounted for about half of all mathematical school tablets found at Nippur. Where metrology was the first part of mathematics education, the second was writing and using place value notation.

3.2.1. Role of the table in Nippur school mathematics

To the scribes, memorizing an ordered set of elementary results (reciprocals, multiplication, etc.) was essential to the mastery of computational algorithms. Proust (2009) notes that whatever the precise role of the tables in the curriculum, the metrological tables supplied the future scribes with two fundamental notions that were new in relation to the lists: the sexagesimal place value system and a correspondence between the measuresments and these positional numbers.

All the measurement exercise worksheets from Nippur show that abstract numbers are used only for calculation, and that metrology tables enabled transformations from concrete measures to abstract numbers (Proust, 2009).

Tablets from the Nippur school show no major differences from other school supplies from nearby Babylonian cities. The same lists and tables have been found in cities like Mari, Susa, Assyria, Ugarit, etc. The wide dissemination of metrological lists and tables in schools and other learning environments indicates that they were the main engine that normalized knowledge in the Old Babylonian period and beyond.

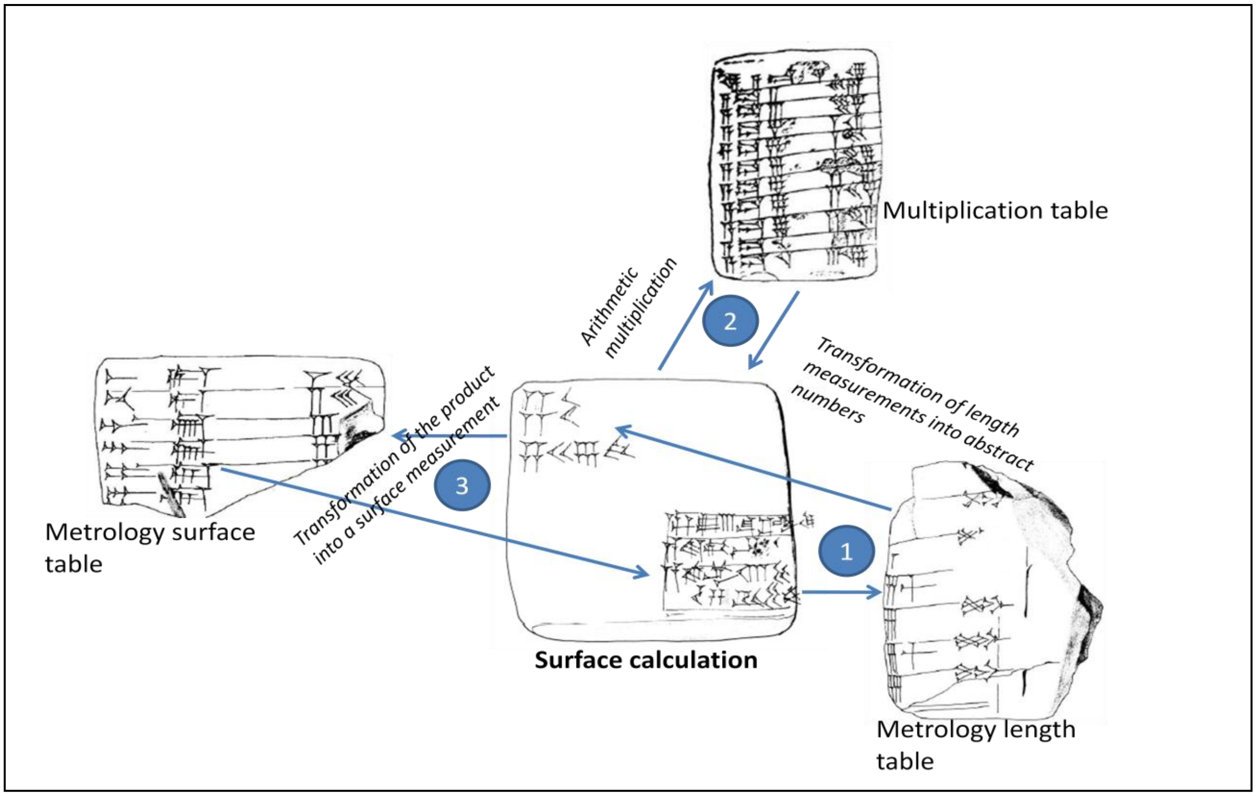


Figure 46. Simulatied area calculation exercise (modified from Proust, 2009, p.38).

Thus, we can conclude that in the curriculum of the scribes of ancient Babylon, the table is presented as a tool for solving problems in the schools of Nippur and also as a tool that helps the standardization of mathematical knowledge among the region's scribal schools.

3.3. Chilean School Curriculum: tables in the statistics topic

In Chile, like many other countries, statistics and probability are included in the mathematics curriculum throughout the school years, from grade 1 to grade 12. Incorporating statistics and probability in the Chilean curriculum at all grade levels began in 2009, causing new demands on teachers who are responsible for the implementation of this component of the mathematics curriculum, a challenge not only for teachers, but also for training institutions.

In our country, most teachers have not had in their initial training studies in statistics, probability, or inference. Furthermore, few of the primary-school teachers in Chile with a mathematics specialty, in their short period of continuing education, have faced the challenge of acquiring the knowledge and skills to meet the demands of teaching statistics and probability. At the same time, high school mathematics teachers have studied on probability theoretically, but not with regard to its teaching. Teachers should know what they teach and how to teach it. Each statistical content item has a different complexity with respect to its mathematical contents; at least at the school level, mathematics and statistics work in different worlds, with different languages and concepts (del Pino & Estrella, 2012; Estrella, 2010).

Moreover, regarding the quality of Chilean education, an OECD report (2010, p.9) advises the country to direct its efforts to improving teacher training at all levels of education, including the quality of initial teacher education programs, the quality of their instructors, and their pedagogical programs. The report states "In general, the government... should identify good practices and give schools the necessary help to disseminate them throughout the school system"

3.3.1. Analysis of the 2012 Chilean curriculum for the data and probability topic of the mathematics course in grades 1 to 6

The national curricular guidelines[16] are the main document at the national level. According to the 2009 General Education Law (LGE), learning objectives are set that all students should achieve, that is, the minimum learning objectives to be achieved by all students and schools in the country at each level and in each subject.

The new General Education Law (Law 20.370, 2009) established new rules regarding curricular instruments to be made available to the school system. This regulation refers, first, to a new structure of the school year, redefining basic education and giving it a duration of six years. The curricular guidelines reflect the need to gradually begin the transition to the new structure, providing a curricular tool that responds to the new composition of the cycle.

Learning Objectives (LO), are objectives that define the expected learning end results for a given subject for each school year and refer to skills, attitudes, and knowledge that seek to

[16] The baseline decree 439/2012 establishes 10 to 20 annual learning objectives for each grade for the mathematics course. Details to help teaching are provided in Decree 1548/2012, for grades 1 to 4, and in Decree 1363/2011 for grades 5 and 6.

promote students' comprehensive development. The formulation of these objectives relates to skills, knowledge, and attitudes and demonstrates clearly and precisely the learning that the student must achieve.

Data and Probability Topic

The curricular guidelines (MOE, 2012, p. 91) state, "This topic seeks for all students to record, classify, and read information provided by tables and graphs, and to begin their understanding of probability. This knowledge will enable them to recognize graphs and tables in their everyday lives. To achieve this learning, they need to know about surveys and questionnaires and implement them through the formulation of relevant questions, based on their experiences and interests, and then record the output and make predictions from them."

Table 6 corresponds to the Data and Probability learning objectives (LO) established in the 2012 curricular guidelines for mathematics for grades 1 to 6 *(Bases Curriculares, 2012).*

Table 6

Summary of Learning Objectives, Data and Probability axis, Grades 1-6

<table>
<tr><th>1°</th><th>2°</th><th>3°</th><th>4°</th><th>5°</th><th>6°</th></tr>
<tr><td>Collect and record data to answer statistical questions about themselves and their environment using blocks, tally charts, and pictograms.</td><td>Collect and record data to answer statistical questions about coins and dice games, using blocks and tally charts, and pictograms.</td><td>Carry out surveys, classify and organize data in tables, and display it in bar graphs.</td><td>Carry out surveys, analyze the data, and compare it with the results of random samples using tables and graphs.</td><td></td><td>Compare the distributions of two groups from random samples using dot plots and stem-leaf diagrams.</td></tr>
<tr><td rowspan="2">Build, read, and interpret pictographs.</td><td rowspan="2">Record the results of random dice and coin games in tables and simple bar graphs.</td><td rowspan="2">Record and order data from random coin and dice games, finding the smallest and largest items and estimating the midpoint between them.</td><td>Read and interpret pictographs and simple bar graphs with scale, and communicate their conclusions.</td><td>Read, interpret and complete tables, graphs simple bar and line and report its findings</td><td>Read and interpret graphs and circular double bar and report its findings</td></tr>
<tr><td colspan="3">Calculate the average of a data set and interpret it in context.</td></tr>
</table>

	Build, read, and interpret pictographs with a scale and simple bar graphs.	Build, read, and interpret pictographs and simple bar graphs with a scale, based on information collected or provided.	Carry out recreational and everyday randomizd experiments, and display the results in tables and graphs manually and/or using educational software.	Describe the possibility of occurrence of an event, using the terms certain - possible - unlikely - impossible.	Conjecture about the trend in results obtained in repetitions of the same experiment with dice, coins, or other, other manipulables and/or using educational software.
		Represent data using scatter plots.		Compare probabilities of different events without calculating them.	
				Use stem-leaf diagrams to represent data from random samples.	

3.3.2. Tables and Graphs in the 2012 Curricular Guidelines for Grades 1 to 6

The representations mentioned in the LO for the data and probability topic[17] for grades 1 to 6 include 7 tables (tally charts and general tables), 17 graphs (pictograms with and without a scale, simple bar graphs without a scale, double bar graphs, line graphs, pie charts), 2 point diagrams (scatter plots), and 2 stem-leaf diagrams.

Since the scatter plot is built on the Cartesian plane, it can be classified as a graph, and as the stem-leaf diagram is constructed with headers and an array of rows and columns, it is possible to classify it as a table.

So the data and probability topic LO express 28 statistical representations, which correspond to 9 tables (32 %) and 19 graphs (68%).

[17] Tally charts 2, General tables 5, Stem-leaf tables (diagrams) 2, Pictograms 5, and Scaled pictograms 1. Simple bar graph 4, Simple scaled bar graph 2, General graphs 2, Double bar graph 1, Scatter Plot 2, Pie charts 1, Line graphs 1.

It is noteworthy that only two of the seven tables explicitly name their type, in this case tally charts. The others five are referred to generally, without specifications, as if they were the same.

The actions that form part of the LO in grades 1 and 2 are collecting and recording data (tally charts). The requirement for grade 3 is to classify and organize data (and display it in graphs). The LO for grade 4 include analyzing and comparing data, for grade 5, reading, interpreting, and completing tables, and for grade 6, comparing two groups' distributions

Table 7

Activities Required in the Guidelines (2012) for Tables and Graphs in the Data and Probability Topic

	Activity 1	Activity 2	Activity 3	Activity 4	Activity 5	Activity 6
Tables	- collect and record in tally charts	- collect and record data in tally charts - record data in tables	- classify and organize data in tables	- analyze and compare data using tables - tabulate data manually or using software	- read, interpret, and complete tables - represent data in stem-leaf diagrams	- compare the distributions of two groups using stem-leaf plots
Graphs	- collect and record data in pictograms - build, read, and interpret pictograms	- collect and record data in pictograms - record data in simple bar graphs - build, read. and interpret scaled pictograms and simple bar graphs	- display data in bar graphs - build, read, and interpret pictograms and simple scaled bar graphs - represent data using point diagrams	- analyze and compare data using graphs - read and interpret pictograms and simple bar graphs - represent data using graphs made by hand or using software	- communicate, read, interpret, and complete simple bar graphs and line graphs	- compare the distributions of two groups using scatter plots - read and interpret double bar graphs and pie charts and communicate the findings

Table 8

Activities representations by Grade Level (1 ° to 6 °)

Representation	Action	1°	2°	3°	4°	5°	6°
Tally Charts	collecting	x	x				
	recording	x	x				
Tables (in general)	recording		x				
	classifying			x			
	organizing			x			
	analyzing				x		
	comparing				x	x	x
	reading					x	
	interpreting					x	
	completing					x	
	representing					x	
Stem-leaf diagrams	representing					x	
	comparing						x
Pictograms	collecting	x	x				
	recording	x	x				
	building	x					
	reading	x					
	interpreting	x					
Scaled pictograms	building			x			
	reading			x	x		
	interpreting			x	x		
Simple bar graphs	recording			x			
	building			x			
	reading			x			
	interpreting			x			
	visualizing			x			
Scaled simple bar graphs	building			x			
	reading			x	x	x	
	interpreting			x	x	x	
	completing					x	
	communicating					x	
Scatter plots	representing			x			

	comparing						x
Graphs (in general)	analyzing				x		
	comparing				x		
	representing				x		
Line graphs	reading					x	
	interpreting					x	
	completing					x	
	communicating					x	
Double bar graphs	reading						x
	interpreting						x
	communicating						x
Pie charts	reading						x
	interpreting						x
	communicating						x

3.3.3. Review of the general perspective of the curricular learning objectives

The introduction to the topic asserts "This topic responds to the need for all students to record, classify, and read data." It would seem to be a low cognitive demand presentation, due to the absence of other demands such as analysis, interpretation, construction, and evaluation. Tables 6, 7, and 8 were built using official curricular guidelines (MOE, 2012), and they show official learning objectives related to data and probability, providing information about requirements established for students.

Based on a reading the official curriculum, these brief paragraphs intend to account for the gaps and omissions observed. Some gaps are clearly related to tables and others not so much, such as the treatment that the official curriculum gives to the concept of a variable, the transition between representations, coordinates, scales, other graphs, and prediction.

Variables

An initial question that arises after reading the LO is, “What is tabulated; what is plotted?” as the official curricular guidelines do not address the concept of a variable or the determination of variable’s categories. The purpose of building tables and graphs is to address the behavior of a variable of interest to obtain information and work towards solving a problem.

Moving between representations

The official curricular guidelines do not include activities referred to changes in register, such as building a pictogram considering data from a table, or moving from a graph to a table, or from one kind of graph to another.

On p. 90 of the curricular guidelines (MOE, 2012), the ability of representation is identified. Specifically, they assert, "Managing a variety of mathematical representations of the same concept and moving seamlessly between them allows students to achieve meaningful learning and develop their ability to think mathematically. During elementary education, it is expected that students learn to use pictorial representations such as diagrams, charts, and graphs for reporting quantities, operations, and relations, and then learn and use the discipline’s specific symbolic language and vocabulary." This quote shows that the transition from one representation to another is explicitly named in the curricular guidelines, but this does not happen in the LO, making the transition between representations an implicit learning objective. Tables are not mentioned in this quote referring to representations.

Coordinates and scales

Other important activities that are implicit in the official LO are the determination of coordinates for building scatter plots, bar graphs, and line graphs, and the concept of a scale.

Although the curricular guidelines (MOE, 2012) mention coordinates in two opportunities, this occurs only in the geometry topic (describing and identifying), and it is also in this topic that the Cartesian plane is named.

Related to the above is the concept of scale, which is a baseline concept for building the gradients of the axes of the graphs, as well as for the units of measurement of the variable. The scale is also related to the value a symbol takes on in a pictogram.

Integrating graphs

In upper levels, grades 5 and 6, to promote making comparisons, the curricular guidelines we can point to triple bar graphs, multiple pictograms, and double or triple line graphs (which appear frequently in the media).

Prediction

A feature of graphs is showing trends, which helps make predictions in real situations of uncertainty. Prediction, although stated in the definition of the data and probability topic, does not appear in the LO of curricular guidelines.

Using Table 3 we can count the activities related to representations by grade level (1 to 6) in the official curriculum, with the result of 17 actions using tables, which specifically consider: collecting, recording, classifying, organizing, analyzing, comparing, reading, interpreting, supplementing, and representing. On the other hand, we find 41 actions related to graphs, which include: collecting, recording, building, reading, interpreting, visualizing, completing, communicating, representing, comparing, and analyzing (see Figure 48).

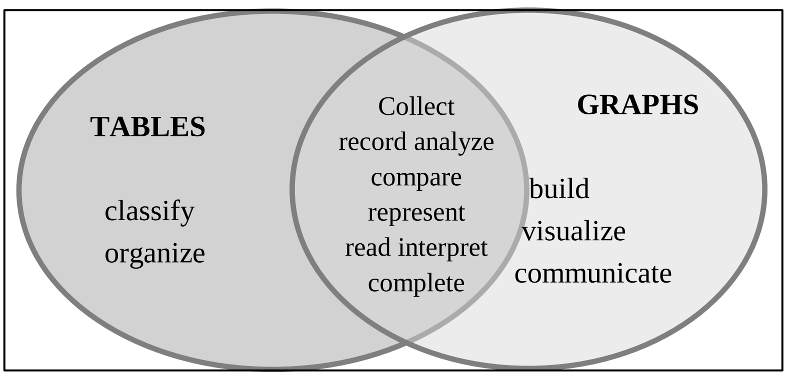


Figure 47 Actions required by the Learning Objectives relate to Tables and Graphs (Prepared according to information contained in the curricular guidelines, MOE, 2012).

As shown in Figure 48, comparing the two types of representations, the activities that are included in connection to graphs but not in connection to tables are building, visualizing, and communicating.

The above analysis opens several questions: Why is implicitly accepted that students can build a table? Is a table appropriate for visualizing the behavior of data? Do tables communicate ideas? In this regard, it is worth noting, in passing, that stem-leaf diagrams are a good example of the tabular format's power to communicate.

3.3.4. Contributions to the curriculum regarding tables

The constituent unit of a table is the list. Teachers can explicitly teach how to build tables from lists.

Operations can be carried out on tables. For example, one can construct simple tables and concatenate them to obtain double tables (in which categories of one variable are joined or two or more variables are brought together) or construct 2 x 2 tables (in which the categories of two variables are crossed).

The curricular guidelines do not include activities where students evaluate the design or the communicability of a table built by themselves or by others. Higher cognitive actions, such as analyzing and interpreting, appear late in the curricular guidelines. In the case of tables, the action of analyzing appears only in grade 4, and interpreting tables appears in grade 5.

3.4. Tables in the school curricula of England, Brazil and Singapore

As stated, we chose the curricula of these countries because they show successful performance in international tests, and their national curricula explicitly include tables.

3.4.1. Curriculum in England.

England incorporated statistics in its primary-school curriculum years ago. In the following, we detail the national curriculum of England for 1999 and 2013.

National Curriculum of England (1999, Handling Data)

Year 1. Handling Data. Pupils should learn to:

- Sort and classify objects showing the criteria used.

Year 2. Handling Data. Pupils should learn to:

- Sort and classify objects with more than one criterion;
- With their data, record results in <u>simple lists, tables,</u> and block graphs to communicate their findings.

Year 3. Handling Data. Pupils should learn to:

- Extract and interpret information present in simple tables and lists.

Year 4. Handling Data. Pupils should learn to:

- Collect discrete data and record it in a frequency table;
- Group and interpret data in equal class intervals and represent it in frequency tables.

Year 5. Handling Data. Pupils should learn to:

- Interpret graphs and diagrams, including pie charts, and draw conclusions.

Year 6. Handling Data. Pupils should learn to:

- Collect and record continuous data;
- Create frequency tables with equal class intervals
- Identify all results in tables or other forms of communication, when dealing with a combination of two experiments

National Curriculum of England (2013, Statistics)

Year 2. Statistics. Pupils should learn to:

- Interpret and construct simple pictograms, tally charts, block diagrams, and simple tables;
- Ask and answer simple questions by counting the number of objects in each category and sorting the categories by number;
- Ask and answer questions about a total and compare categorical data.

Year 3. Statistics. Pupils should learn to:

- Interpret and present data using bar graphs, pictographs, and tables;
- Answer one-step and two-step questions [e.g., "How many more?" and "How many fewer?] using information presented in scaled bar graphs, pictographs and tables.

Year 4. Statistics. Pupils should learn to

- Interpret and present discrete and continuous data using appropriate graphical methods, including bar graphs and time graphs;

- Solve comparison, sum, and difference problems using information provided by bar graphs, pictographs, tables, and other graphics.

Year 5. Statistics. Pupils should learn to:

- Solve comparison, sum and difference problems using information presented in line graphs;
- Complete, read, and interpret data in tables, including timetables.

Year 6. Statistics. Pupils should learn to:

- Interpret and construct pie charts and line graphs and use them to solve problems;
- Calculate and interpret the arithmetic mean.

3.4.2. Brazilian Curriculum

The Brazilian curriculum contains interesting proposals like those from England. Regarding statistics, the goal is for students to construct procedures to collect, organize, communicate, and interpret data using tables, graphs, and other representations that appear frequently in their daily life.

Brazilian Curriculum (1997, Information Processing)

First Cycle

- Reading and interpreting information contained in images;
- Collecting and organizing information;
- Making personal records for communicating collected information;
- Exploring the function of numbers as a code for organizing information (bus lines, telephone numbers, vehicles license plates, identity records, libraries, clothing, shoes);
- Interpreting and making lists, simple tables, crosstabs, and bar graphs to communicate the information obtained;
- Producing written texts based on the interpretation of graphs and tables.

Second Cycle

- Collecting, organizing, and describing data:
- Reading and interpreting organized data (through lists, tables, diagrams, and graphs) and constructing these representations;
- Interpreting data from tables and graphs to identify predictable characteristics or random events;
- Producing written texts from the interpretation of graphs and tables, constructing graphs and tables based on the information contained in journalistic, scientific, or other texts;
- Obtaining and interpreting the arithmetic mean;
- Exploring the idea of probability in simple problem situations, identifying possible events, certain events, and "random" situations;
- Using given information to evaluate probabilities;
- Identifying possible ways to combine elements of a collection and record them using personal strategies.

3.4.3. Singaporean Curriculum

Singapore is internationally recognized as a country that is successful in international mathematics tests. It has become a model for improving the curriculum in several countries. Below we concisely show some aspects of the statistics topic of the curriculum.

Singaporean Curriculum. Syllabus (2007, Data Analysis)

Level 1: collecting and organizing data, building pictograms, using drawings or symbols to represent an object, reading and interpreting horizontal and vertical pictographs. Excludes scaled pictograms;

Level 2: building scaled pictograms, reading and interpreting scaled pictograms, solving problems using information in pictograms. Excludes the use of incomplete drawings or symbols [such as half a tree].

Level 3: reading and interpreting bar graphs horizontally and vertically, reading scales, completing bar graphs from given data, solving problems using information in bar graphs.

Level 4: Completing a table from given data, reading and interpreting tables, and solving problems using information offered in tables

Level 5: completing a table/graph from given data, reading and interpreting tables/graphs, and solving problems using information presented in tables/graphs (bar and line graphs)

3.4.4. Comments

The curriculum for England for 1999, with regard to tables in the early years of schooling, incorporates the concepts of lists and tables for grades 2 and 3, and frequency tables for grades 4 and 6. In the new version, 2011, tables appear in grades 2 to 5, and differences are established between tally charts, simple tables, and tables in general. The curriculum differentiates the use of discrete and continuous data and the use of frequency tables, including data grouped by intervals, from 4th grade onward. Changes in the curricular made in September 2013 can be seen in the formulation of questions, in building tally charts and simple tables in Grade 2, and in the concepts of categories and categorical data. In grade 4, it changes the emphasis, proposing solving problems using information offered in tables. Such emphasis is sustained through grade 5, in which completing, reading, and interpreting information given in tables are added.

The Brazilian curriculum, formulated 16 years ago, has pioneered statistics ideas, such as making lists and simple and double entry tables, as well as producing written interpretive texts from tables. This curriculum seeks for students to be autonomous and to be able to read and interpret organized data presented in lists, tables, diagrams, and graphs, and also to be capable of constructing such representations.

The Singaporean curriculum starts with pictograms and does not consider the table or list in the first three grades. Questions arise: How do students build graphs? Where do students organize their data? Despite this absence, the place given to tables from grades 4 and 5 onward is notable -students are asked to read, construct, and interpret tables-, and although tables are not considered in grade 6, they are in grade 7, which again explicitly calls for the "construction and interpretation of tables".

4. TABLE IN INTERNATIONAL ASSESSMENTS

4.1. Introduction

After revising some national curricula, we realize that few countries include objectives and activities related to learning tables; exceptions are countries such as England, Japan and Brazil. It is common to see tables appear in textbooks as a tool but not as an object of study.

In general, textbooks do not include activities for learning to read or concatenate tables, as presented in section 1.5. We might assume that curricula and textbook authors are convinced that students learn to read tables in the pre-school stage or spontaneously, an affirmation that we object to, based on the evidence provided by international results of the TIMSS Mathematics test for 4th grade, from the years 2003, 2007, and 2011, for example.

In this section we show items released from the TIMSS test and its performance levels, low, medium, high, advanced, and upper advanced, indicating the percentage of correct responses at the international level and, where appropriate, nationally (in Chile) (detail of reference levels for grade 4 and 8 are shown in Appendix IV.1).

4.2. Tables in the TIMSS test

54% of 4th grade students who were asked the question M051109[18] -Figure 49- answered correctly, compared to only 34% of Chilean students.

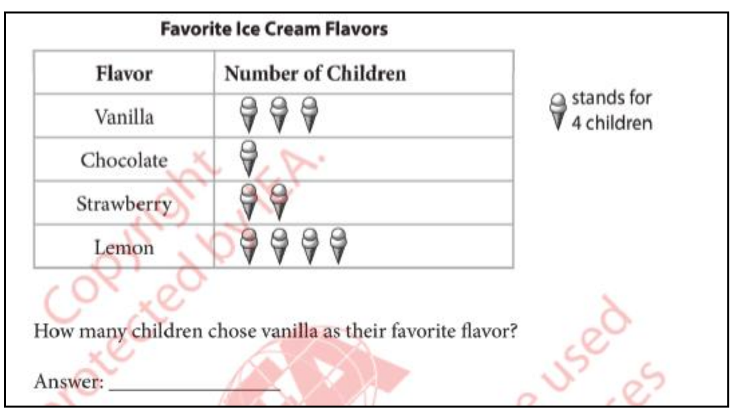


Figure 48 . Pictogram-type table item, Intermediate level TIMSS (2011).

[18] Corresponds to the ID code of the TIMSS item

It seems clear that reading tables, in this case associated with a pictogram, requires a teaching process, since the intuitive reading, without considering the header and the meaning of the figure, is misleading.

According to student performance, results of TIMSS assessments have allowed for characterizing the levels of complexity of the test items. For example, it is said that a student's performance level is low if it is around 400 points (+/- 10 points) on the test, significantly below the average. We say that a low-performing student is able to solve a particular item if between 50 and 65% of these low-level students are able to answer the item in question. A student with a low level of performance is able to read and complete information in bar graphs and simple tables, for example, he or she is able to respond to item M031133 (see Figure 50).

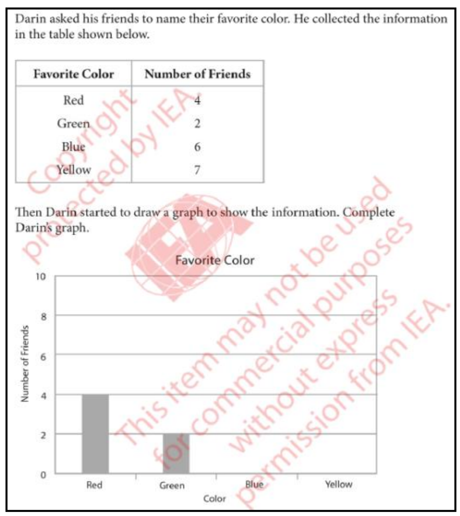


Figure 49 Table and graph item, Low level, TIMSS (2011).

A student with an intermediate level of performance, who gets 475 +/- 10 points on the TIMSS test, is able to complete a two-by-two table to summarize information, e.g., he or she responds correctly to M031240 (see Figure 51).

There were 5 children at the park. Some were wearing hats and some were not.

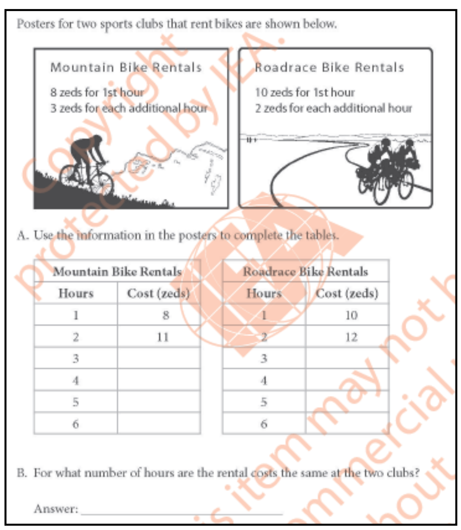
Girls	Boys
Maria was wearing a hat	Peter was wearing a hat
Megan was not wearing a hat	Chan was not wearing a hat
Mandy was not wearing a hat	

Complete the table to show the number of boys and girls that were wearing hats and were not wearing hats.

	Hat	No hat
Boys		
Girls		

Figure 50 2 x 2 table item, Intermediate level, TIMSS (2007).

A student with a high level of performance, who gets 550 +/- 10 points on the TIMSS test, is able to read and interpret data from two tables to answer a question. For example, he or she correctly answers the question M031242B (see Figure 52).


Posters for two sports clubs that rent bikes are shown below.

A. Use the information in the posters to complete the tables.

Mountain Bike Rentals	
Hours	Cost (zeds)
1	8
2	11
3	
4	
5	
6	

Roadrace Bike Rentals	
Hours	Cost (zeds)
1	10
2	12
3	
4	
5	
6	

B. For what number of hours are the rental costs the same at the two clubs?

Answer: ______________

Figure 51. Two-table item, High level, TIMSS (2007).

A student with an advanced level of performance, who gets 625 +/- 10 points on the TIMSS test, is able to organize data and complete a tally chart to represent the data. For example, he or she correctly answers the question M031134, see Figure 53.

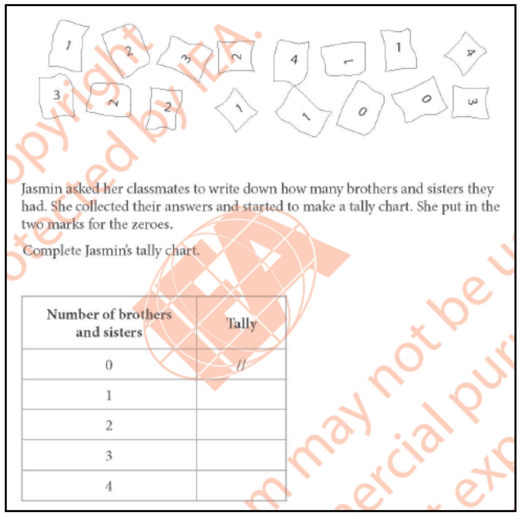


Figure 52 Tally chart completion item, Advanced level, TIMSS (2007).

The test for grade 8 offers an item with a performance level of upper advanced; this level, typical of some questions, serves to further characterize the skills and knowledge of students showing better performance (advanced level), but is not a level of student performance. This item (M052503) is not related to tables but shown as an example of a question that asks students to reason with information in two representations and justify the answer (see Figure 54).

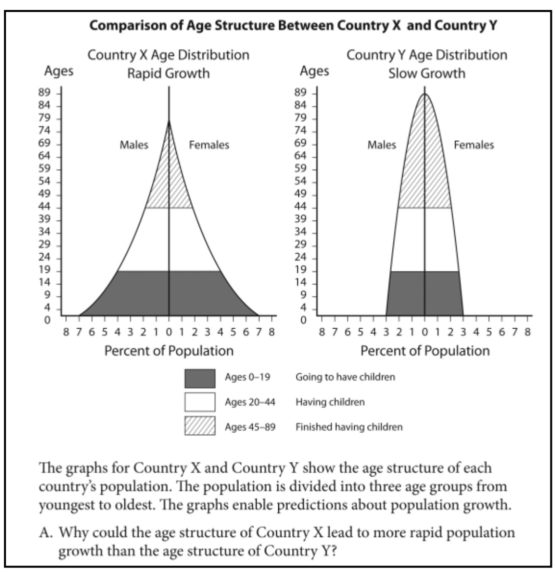


Figure 53 . Two-graph item, upper advanced level, TIMSS (2011).

Several errors in handling tables that elementary students make persist in secondary education, although they could have been corrected early. Espinel and Antequerra (2009) presented secondary students with four activities from the 2003 PISA test that use tables to resolve everyday situations. The study shows the errors in handling tables made by students and points out aspects that training can solve.

4.3. Review of TIMSS items

Tables 9 and 10 summarize the Chilean students' performance on the 2011 TIMSS assessment according to content areas and cognitive domain. The math test has a scale of 0 to 1000 points with the international average calculated at 500 points.

Table 9
Chilean Performance on TIMSS-2011 Test (grades 4 and 8) compared to the average, by content areas

Content area	*Average*	Number	Geometry and Measurement	Data	--
Chile Grade 4	462	0	–6	+4	--
Content area	*Average*	Number	Algebra	Geometry	Data and Probability
Chile Grade 8	416	–4	–14	+3	+9

Table 10
Chilean Performance on TIMSS-2011 Test (grades 4 and 8) compared to the average by cognitive domain

Cognitive Domain	*Average*	Knowledge	Application	Reasoning
Chile Grade 4	462	–6	+1	+7
Chile Grade 8	416	–11	+9	+5

At the country level, Chile shows a low performance in Mathematics for grade 4 and grade 8 (about 400 points and under 474).

In the TIMSS definition of the performance levels of students in grade 4 for statistics, tables are explicitly included at all levels. However, for grade 8, they are explicitly included in 2 of the 4 levels.

Only with respect to the statistics items for Grade 4:

- At the low level, students have some basic math skills, and can read and complete information in simple bar graphs and tables;

- At the intermediate level, students apply basic knowledge in simple situations, and can interpret information in bar graphs, pictograms, and tables to solve simple problems;

- At the high level, students apply their knowledge and understanding to solve problems, can interpret and use data in tables and graphs to solve problems, and can use information in pictograms to complete bar graphs;

- At the advanced level, students apply their knowledge and understanding in a variety of relatively complex situations and can explain their reasoning and also manage to draw a conclusion from data in a table and justify it.

Only with respect to statistics items for Grade 8:

- At the low level, students have some knowledge of whole numbers and decimals and simple graph operations, and can relate tables to bar graphs and pictograms and read simple line graphs;

- At the intermediate level, students apply basic skills in a variety of situations, and can read, interpret, and construct graphs and tables. They also have basic knowledge of probability;

- At the high level, students apply their knowledge and understanding in a variety of relatively complex situations and analyze and manage data in a variety of graphs;

- At the advanced level, students show reasoning skills, establish conclusions, make generalizations, and solve linear equations, and are capable of reasoning with data from various sources or unfamiliar representations to solve problems that involve multiple steps.

Table 11 presents a summary of some characteristics of 12 items regarding tables released from the international TIMSS test (2003, 2007, and 2011). It displays each item, its performance level (low, intermediate, high, or advanced), its content domain (numbers, algebra -if applicable-, geometry, and statistics), and the average number of correct answers internationally and nationally (if applicable). Appendix IV.2 contains a similar table, with the difference that the first column provides an image of each item.

Table 11
12 TIMSS items and their characteristics

TIMSS items regarding tables	Level of performance	Content Domain	Cognitive Domain	International average	Chilean average (year)
"Favorite ice cream flavors," open-ended question about a pictogram-type table. The answer requires numerical conversion of the symbolic icon.	Intermediate M051109	Data representation	Knowledge	54%	34% (2011)
"Favorite Colors" is an open-ended question that asks students to complete a bar graph from a table of given data. The answer requires drawing two bars whose height is given by the frequency.	Low M031133	Data representation	Application	73%	69% (2011)
"Number of brothers" is an open-ended question that asks students to organize data and complete a tally chart. The answer requires completing a mark for each item counted (numbers, i.e. frequency, would count as an error).	Advanced M031134	Data representation	Application	27,7%	n/p (2007)
"2 x 2 Table" is an open-ended question that asks students to make a count of 5 data entries about two variables , gender and possession of a hat, and complete a table with the count using headers built by the student (the two variable categories)	Intermediate M031240	Data representation	Application	48,1%	n/p (2007)
"Rent a bike" is an open-ended item that asks students to complete 2 tables for 2 advertisements. The answer requires a constant sum to complete a column for each table (given in the advertisement)	High M031242	Data representation	Knowledge	52,3%	n/p (2007)
"Trees in a park" is a multiple choice item. Responding includes reading data from the table and associating it with pie charts (areas of sections of a circle)	Intermediate M031045	Data representation	Reasoning	61,1%	n/p (2007)
"Examination scores" is a multiple choice question. It corresponds to a frequency table that contains a table counting brands. The question requires reading the table, understanding the relationship "greater than", and adding some frequencies	High M012037	Statistics	Reasoning	48,8%	33,1% 8vo (2003)
"Number of houses" is a multiple choice question that asks students to complete a pictogram-type table. The answer requires making a numerical equivalence with the icon	Intermediate M031172	Data representation	Application	56,7%	n/p (2007)
"Number of blocks" is a multiple choice question that asks students to convert the numerical value of a pictogram-type symbolic icon into a table. The answer requires dividing by the numerical equivalence of icon.	High M041186	Data representation	Knowledge	49,7%	n/p (2007)
"Speed of cars passing by a college" is an open-ended question. Given a table of marks, students are asked to complete a bar graph. The answer requires using a given variable category, making a count of the marks, and then drawing the bars with the height given by the frequency found (frequency is marked on the Y axis , but does not show up, so the location must be estimated in order to draw the bars.	Advanced M041275	Data representation	Application	50,7	n/p (2007)
"Phone Plans" is an open-ended question. It requires reading a table of 5 columns and 3 rows containing information about fixed and variable prices depending on the time of day. The answer requires completion and explanation with reference to free minutes and a lower monthly fee.	Advanced M032762	Statistics	Reasoning	21,4%	12,1% 8vo (2003)
"Sudents' favorite sports" is an open-ended question that asks students to read a frequency table and draw a pie chart with the circumference divided in eighths. The answer should correctly divide and label the graph in 8 sections (1-2-2-3)	High M042207	Data and probability	Application	46,7%	43,7% (2011)

In summary, TIMSS goals in the data domain, for grade 4, focus on reading and representing data, and in grade 8, emphasize data interpretation and the foundations of probability.

Based on the analysis of the items listed in Table 11 and section 4.2 of this chapter, it appears that the TIMSS questions are mainly focused on mathematical procedures and are not representative of statistical thinking. For example, several items ask students to determine the numerical value corresponding to iconic expressions by means of counting or addition. Exceptions are advanced and upper-advanced performance level items for grade 8 (M032762 and M052503, respectively), whose situations are presented in a context that gives meaning to the justifications for the answer relating variable(s) with the variable categories involved.

4.4. Studies of cognitive processes associated with representations

While the literature on education recognizes the importance of studying cognitive processes associated with understanding graphic and tabular representations, research has privileged the study related to graphs. For example, research by Friel, Curcio, and Bright (2001), based on graph and table understanding, focused on tables linked to graph representations, considering tables only as an intermediate step to the creation of graphs, a "transition tool" (p. 128)[19]. If we consider that tables help us observe regularities and therefore reflect on the data, the authors' assertion regarding the use of tables that they considered lets us conclude that their taxonomy of understanding was built only for graphs and should not be used indistinctly for tables. These authors add that it is not clear how tables have been incorporated into the work of organizing and describing data (2001).

Graph comprehension has been studied by Aoyama (2007), Baillé and Vallerie (1993), Bertin and Barbut (1967), Curcio (1987, 1989), and Friel, et al. (2001). Curcio (1989) defines three types of understanding of graphs. Later, Shaughnessy (1996, 2007)[20] identifies a fourth type, extending the previous levels. The most basic level of understanding, "reading data" requires a local and specific action such as the literal reading of the graph, and is concerned only with the

[19] Tables may also be used for organizing information as an intermediate step to creating graph representations.

[20] There are some documents in Spanish that say Friel et al. are the creators of the fourth taxonomic level of graph understanding, but Shaughnessy (2007, p. 989) references his own work at this level, Shaughnessy, Garfield and Greer (1996). This work is referenced in Friel et al., 2001.

explicitly represented facts. Therefore, no interpretation of the information contained therein is made. "Reading between the data" involves comparing and interpreting data values, integrating the data in the graph, finding relationships between quantities, and applying simple mathematical methods to the data, understanding both the basic structure of the graph as well as the relationships contained therein. "Reading beyond the data" involves extrapolating data, predicting and inferring from the data information that is only implicitly present in the graph, and requires knowing the context in which the data is presented. Finally, "reading behind the data" refers to looking critically at the use of the graph and connecting the data to the context for an in-depth analysis and causal reasoning based on subject knowledge and experience; it includes examining the quality of the data and its collection methodology, suggesting a possible explanation, and developing alternative models and graphic representations.

Overall, examples presented on student performance on table items from the TIMSS test reveal difficulties in learning tables, and open an area of study about the types of tasks, the variety of readings, and the cognitive resources that are put into play when using and eventually building tables.

Arteaga et al. (2011, p. 60) describe the taxonomy of graph understanding and argue that it can also be considered for reading tables. While graphs and tables are linked, there are differences in their structures and the purposes of using them, an issue that underlies the search for a taxonomy specific to tables. In principle, the difference between the structure of a table and the structure of a graph might lead to different taxonomies for graph and table understanding. This position is consistent with the findings of Gabucio et al. (2010), which identified a fourth category of tabular understanding, different from the three categories of graph understanding established by Curcio (1989). Gabucio et al. (2010), experimenting with 5th grade students, identified a gradient in table interpretation tasks, specifically, direct reading of data, understanding of the table structure, inference from data, and overall interpretation.

The literature review shows that the topic is open, and that although taxonomies have been postulated for dealing with graphs, there are none for tables. Research on representations has focused on graphs and considers tables as a tool in the construction phase of graphs. The intended use or treatment of the tables themselves, which may differ from that of graphs, has not been taken into account. Graphs privilege images and reveal trends, whereas tables specialize in specificity and organization.

4.5. Characterization of the roles of individuals faced with tasks related to tables

Hurst (2000) identifies a model for describing tables. Among the dimensions he considers is the table's structure and its use by the reader. Regarding the second aspect mentioned, this is the role of the reader in tasks related to tables: in the tasks of building a table to organize data and analyzing a table to extract data or to identify regularities in it, subjects assume different roles, eventually putting into play cognitive processes of different levels.

Some studies, for example Wang (1996), show that users interact with existing tables using at least three cognitive processes: understanding, searching, and interpreting. The understanding process involves understanding the organization of a table in order to capture the logical structure underlying it, the search process is related to locating relevant information within the table, and the interpreting process involves comparing and is necessary for responding to specific questions after the relevant information has been obtained.

Based on these three cognitive processes proposed -understanding, searching, and interpreting-, a table should be organized so that the underlying logical structure becomes obvious and table items can be located and interpreted more easily.

After reviewing the TIMSS items and carrying out an epistemological study of the uses of tables, we identified six categories of roles that individuals assume when faced with tasks associated with tables.

An initial study allowed us to detail the subject's roles, as presented in Figure 55 (Estrella, Mena-Lorca & Olfos, 2014)[21].

The categories studied are *recorder, searcher, completer, builder, interpreter*, and *evaluator*.

[21] Note: in Figure 55, the meaning of: (a) is a more basic action that is simpler to explain, while (r) is an action that requires arguments, justifications, and reasoning in order to explain the answers.

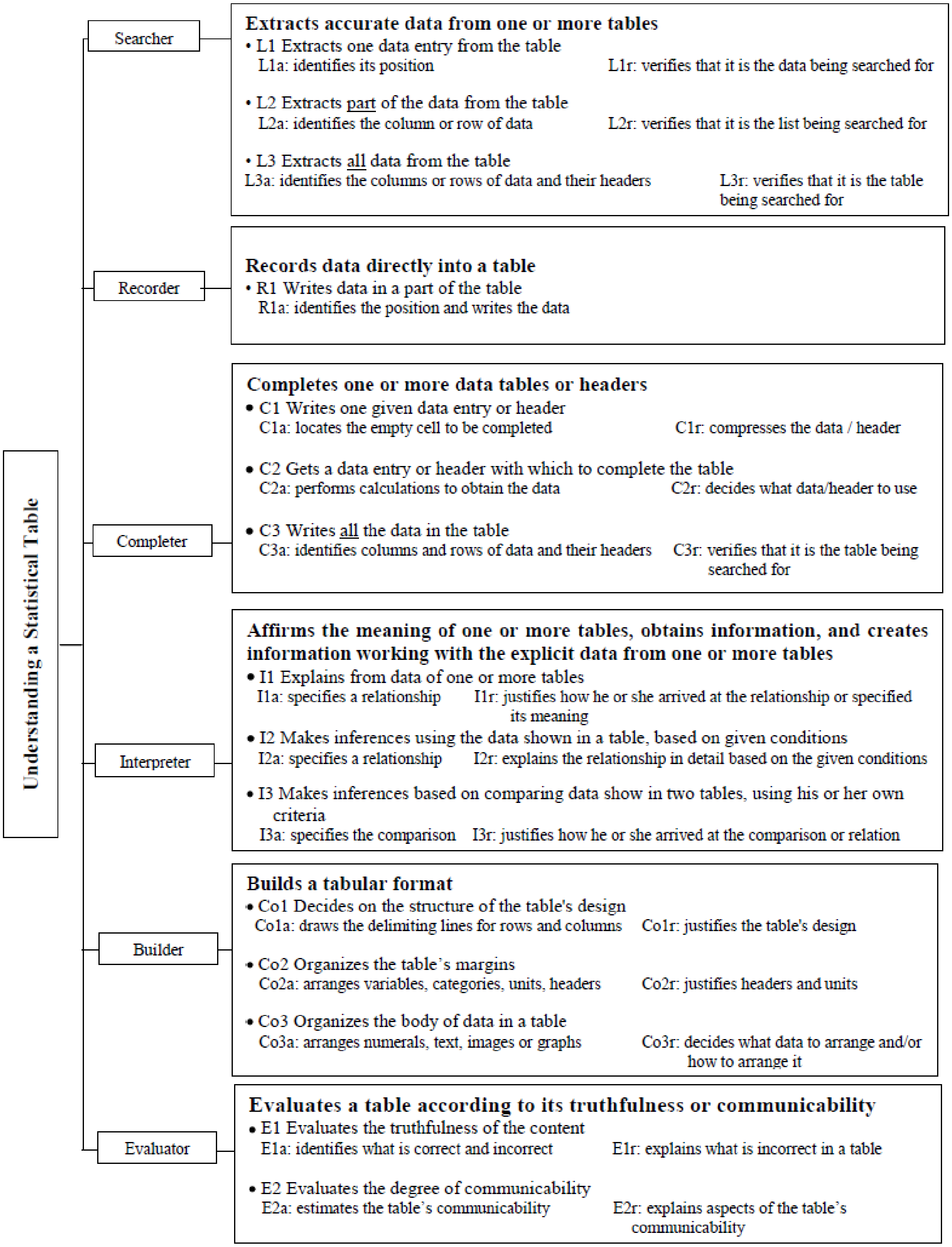


Figure 54 Comprehension framework for statistical tables (Estrella, Mena-Lorca & Olfos, 2014).

Table 12 contains the characterization of six categories of subject roles according to the purposes of corresponding tasks. The first three categories of subject roles –recorder, searcher, completer- are linked to lesser cognitive demands than the categories builder, interpreter and evaluator.

Table 12
Subject roles according to the purpose of the table task

	Roles					
Category	**search**	**record**	**complete**	**build**	**interpret**	**evaluate**
Task	Extraction of given data and/or categories to obtain information. .	Maintaining data records to eventually obtain information. Atemporal memory.	Completion and compression of relationships between data and class categories.	Communication through functional organizational layout showing a partition and its categories and classes.	Comparison and identification of relationships between categories. Explanation of relationships and the information obtained.	Assessment of the usefulness of the layout chosen for the table, with respect to understanding and clarity.
Action specific to the Task	Read the categories of the lateral and/or superior headers Count and/or recognize marks, numbers, linguistic and non-linguistic units.	Record the data of interest with linguistic or non-linguistic units and/or numbers, principally in the body of data.	Complete with category names in lateral and/or superior headers. Complete with marks or numbers according to a count. Read the headers of empty cells. Compare.	Make explicit in rows and columns, and combinations thereof the lateral and superior headers, and the body of data in cells. Make explicit each category and write marks, numbers, or linguistic or non-linguistic units	Relate data and categories to explain and compare. Identify relationships and/or patterns that arise from the table.	Monitor clarity and understanding. Make judgments about the table.

Implicit Action specific to the tak	Visually browse the table, locating and relating non-empty cells with headers. Search and recognize.	Read, compare, and classify to record. Focus and repeat.	Locate empty cells and relate them to headers. Identify, compare, and decide	Determine spaces to complete. Decide and design.	Explore, compare, relate, evaluate, and discard or no.	Analyze, monitor, and evaluate. Search for errors or difficulties.
Explicit Action specific to the task	Search and point out physically or orally. Locate and identify.	Write or draw marks, numbers, or linguists or non-linguistic units, given a fixed category of either a lateral or superior header.	Visually browse, locating empty cells to complete in relation to headers. Search, identify, compare, interpret, and write.	Plot horizontal and vertical segments, which create cells. Decide on and write compressed linguistic in the lateral and superior headers.	Justify relationships and comparisons. Explain and make explicit. Provide answers and/or predictions.	Express the evaluation of the clarity and understanding of the format and relationships.

The transition from the tasks associated with tables to subjects' roles links the tasks to cognitive processes and consequently to levels of table understanding.

5. CHAPTER CONCLUSIONS

The goal of this chapter was to outline a didactic for the table object, in terms of its production, dissemination, and learning, taking into account the institutions and activities that use it. To do so, we considered tables in computer science, in statistics, and their place in school mathematics curricula and in international assessments.

One contribution to teaching tables at the school-level is the generic table model provided in section 1.4, which combines structural location with statistical content.

Additionally, we have treated the table as a knowledge object, putting it into practice in a process of concatenation of simple tables extracted from a 3rd grade mathematics textbook. This task is explicit evidence of the possibility of teaching table operations in primary school.

We have presented two perspectives on tables as a mathematical knowledge object, which relate to the epistemological study carried out:

1. The generic table model and the identification of the composition of tables, which allows us to look at the table as a knowledge object that has an explicit definition, properties, and operations.
2. The Nippur findings allow us to see the table as a knowledge object that comes into existence when scribes, in their role as agents in the education system, become aware of it and insert it into the system of objects to be taught, or objects to be learned, due to its useful nature in the economy of the didactic system they face in their schools.

Among existing curricula, the Chilean curriculum presents important content gaps, such as the concept of variable, treatment of both scales and coordinates, transitions between representations, and prediction. After analyzing the explicit activities in the Chilean curricular guidelines, it was found that there are no table construction tasks and that tasks related to visualizing and communicating are not made explicit. The absence of cognitive activities like "build - display - communicate" in the curriculum is important for math and science learning and for the development of school scientific research, as well as for student performance on international tests. The Chilean curriculum seems to distance itself from the ideas of Exploratory Data Analysis (Tukey, 1977), which respond to a general movement in statistics education that promotes and values the use of representations as an analysis tool and not just as a means of communication.

Curricula in England and Brazil allow us to observe a proposal that takes into account the findings of our epistemological study regarding lists as precursors to tables and lists as the basic unit of tables. Nisbet (1998) reports, in his study of categorical data representations with prospective teachers, the results of 11 different types of data representations: from lists (grouped and ungrouped) to tables and other graphs. It would seem that a natural sequence is to pass through lists first in order to configure the conceptualization of tables.

Finally, after studying TIMSS items, the role of subjects facing table tasks is outlined and this allows us to delineate difficulties and levels of cognitive demands. These questions open an area of research on the emergence of tables in early grade levels and inquiry into a taxonomy of table understanding, a task which we will focus on in the next chapter.

CHAPTER V

Four Studies on Tables

Chapter Summary

Introduction to the four studies

Introduction to the four studies

This chapter presents four studies that give us an approximation of the understanding of table learning at school level. It begins with an analysis of the evidence that emerges from student productions and instructor management of task demands, in a situation of data analysis. We continue with the characterization of the types of interpretation that dealing with tables demands, proposing categories to create a hierarchy of understanding specific to tables, which finally will be tested.

Studies 1 and 2 are based on a case study of a proposal for a data analysis lesson. The first study addresses the cognitive aspects of learning and attempts to respond to the question, “How does the notion of tables emerge in children?" Specifically, the question, “How do children create meaning from data?” contributes to elucidating the process of data transnumeration, that is, achieving greater understanding in the process of moving from data to a table representation. The second study is also didactic and addresses the question, "How can the teacher maintain the task’s level of cognitive demand?" which seeks to characterize a lesson apt for the development of statistical reasoning through data analysis and the use of graph and table representations.

Studies 3 and 4 are of a cognitive nature and are based in the creation of a taxonomy of table understanding that seeks to respond to the question, “What are the cognitive demands of the tasks associated with tables?” The third study develops and uses a taxonomy of table understanding based on the structure of the table and the role of the subject according to the purpose of the associated task; specifically, it identifies the cognitive processes that are activated with frequency tables at school level. The fourth study contrasts the table taxonomy developed with an existing taxonomy for the understanding of graphs, and seeks to determine if they share a common orientation.

As we are interested in understanding the phenomenon being studied, obtaining useful descriptions and conclusions, and critically observing the findings, we have implemented triangulation in our research studies as a way of addressing one object of study from various perspectives of contrast and confrontation. In the first three studies we used triangulation, in comparing data sources, in the contraposition of perspectives of different researchers, in investigating based on different theories, in using different methods, and based on the agents being researched.

STUDY 1: INITIAL CONCEPTUALIZATION OF THE TABLE

1. SUMMARY

This study addresses the conceptualization of the table and its teaching possibilities at school level. The theoretical framework adopted is the theory of conceptual fields, which allows for a cognitive analysis of conceptualization in learning. From a theoretical perspective, Vergnaud affirms that mathematics is needed to characterize with minimum ambiguity the knowledge contained in ordinary mathematical competencies and highlights the fact that this knowledge, although intuitive and implicit, should not hide the need for mathematical concepts and theorems for analyzing it. The study explores primary-school children's comprehension and how they make sense of statistical frequency tables by examining their productions when they are faced with a data analysis situation. From a qualitative research standpoint, the mathematical relations underlying the operations students use to resolve a situation were analyzed. Based on the theoretical framework, a set of mathematical terms was compiled (partition, cardinality, equivalence relation, set) to describe the students' productions. The principal result of this study is the identification of different levels of conceptualization of the table -for students in the same age group- from reduced, uncounted, iconic lists, to tables with text, counting, and marginal totals. The different levels of conceptualization of the table encountered are indicators of statistical thinking. Of a total of 80 third-grade students, only two students created -without prior instruction- what is recognized as a frequency table.

2. ANALYSIS OF PRODUCTIONS USING VERGNAUD'S THEORY OF CONCEPTUAL FIELDS

2.1. Introduction

The existing literature in mathematics education and statistics education, in general, does not address table learning in children. Some exceptions are the works of Brizuela and Alvarado (2010), Gabucio et al. (2010), Marti (2009), Marti et al. (2010), Ben-Zvi (2005), Brizuela and Lara-Roth (2002). The research of Brizuela and Lara-Roth (op. cit.) showed that 7-year-old students could use tables to resolve problems of an algebraic nature. Although these students had not received direct instruction in the use and configuration of tables, they were able to utilize tables to work on a problem. The tables in Brizuela and Lara-Roth's study were produced without a structure being imposed on the children, which is an element shared by the present study. Marti (2009, p. 12) also explored the use of tables by students and found that, “The process of table construction can change the subjects’ a priori knowledge.”

In our research, we have considered a list as that which has a one-dimensional character, including enumeration and/or classification -of things, quantities, etc.- and a disposition in columns (vertical reading) or in rows (horizontal reading), does not posses headers, and whose components are separated by spaces and/or punctuation. In contrast, we consider a table to be that which occupies two dimensions -vertical and horizontal-, and is made up of headers and a corpus of data, localized in rows, columns, and cells, which may or may not have marked lines. Additionally, a frequency table is considered in its most basic aspect as a rectangular arrangement with a structure that includes a set of rows and columns (as previously defined) that allows for representing data corresponding to a variable -characteristic of the phenomenon being studied- in an ordered and summarized manner, and facilitates understanding of the behavior of the data and comprehension of the information that can be extracted. In summary, a frequency table is considered to be that which possesses the previous characteristics, explicitly presents the count of each element of the lesson (subcategory), and, eventually, their marginal totals (category count).

Theoretical perspective for analyzing tasks

This study shows the process of table construction in situ and investigates prior conceptions and intuitive strategies prior to teaching. We adopt Vergnaud's theory of conceptual fields (TCF) as a theoretical framework in order to investigate the structure of the concept created and describe its levels of conceptualization (the transformation of a concept in the form it is taught to the form in which it is conceived by the learner).

The analysis of conceptualization begins with *schemes* and continues with an analysis of the activity. The analysis of the conceptualization of tables must be carried out based on the written productions of the students when they resolve the problem. Although a *scheme* is not a behavior, it has the function of generating activity and behavior in a situation. As such, while we cannot directly access the mental schema that direct the students’ responses in the situation, it is possible to study them through an analysis of the actions and gestures executed, in particular the operational invariants that make the schema work.

An operational invariant is considered to be implicit mathematical knowledge within the *scheme*, which directs the student's recognition of the pertinent elements of the situation and the apprehension of information regarding the situation. In our theoretical framework, the concept of a table, C_{table}(S, OI, SR), includes the situation (S), determined by the task that gives meaning to the concept table, tasks that include ordering, classifying, categorizing, comparing, or registering; the meaning (OI) given by the invariants upon which the division in classes designed to provide order and relations is based; and the signifier (SR), the integration of symbols and language in the visual network, the grid -marked or not- spatially recognizable in the plane, and its properties.

The concept of transnumeration is also considered in its relation to statistics education, and it is pertinent to this study in as much as productions are analyzed in relation to a situation of data analysis that addresses a curricular goal in statistics.
The questions that guide this study are:

> What characteristics of school-level conceptualization of a frequency table can be inferred based on students' work in resolving problems in a data analysis situation? How does transnumeration present itself in this process?

2.2. Methodology

This study uses a qualitative focus to study students' knowledge about the concept table based on experimental data obtained in the resolution of the situation. The methodological design is shown in figure 56

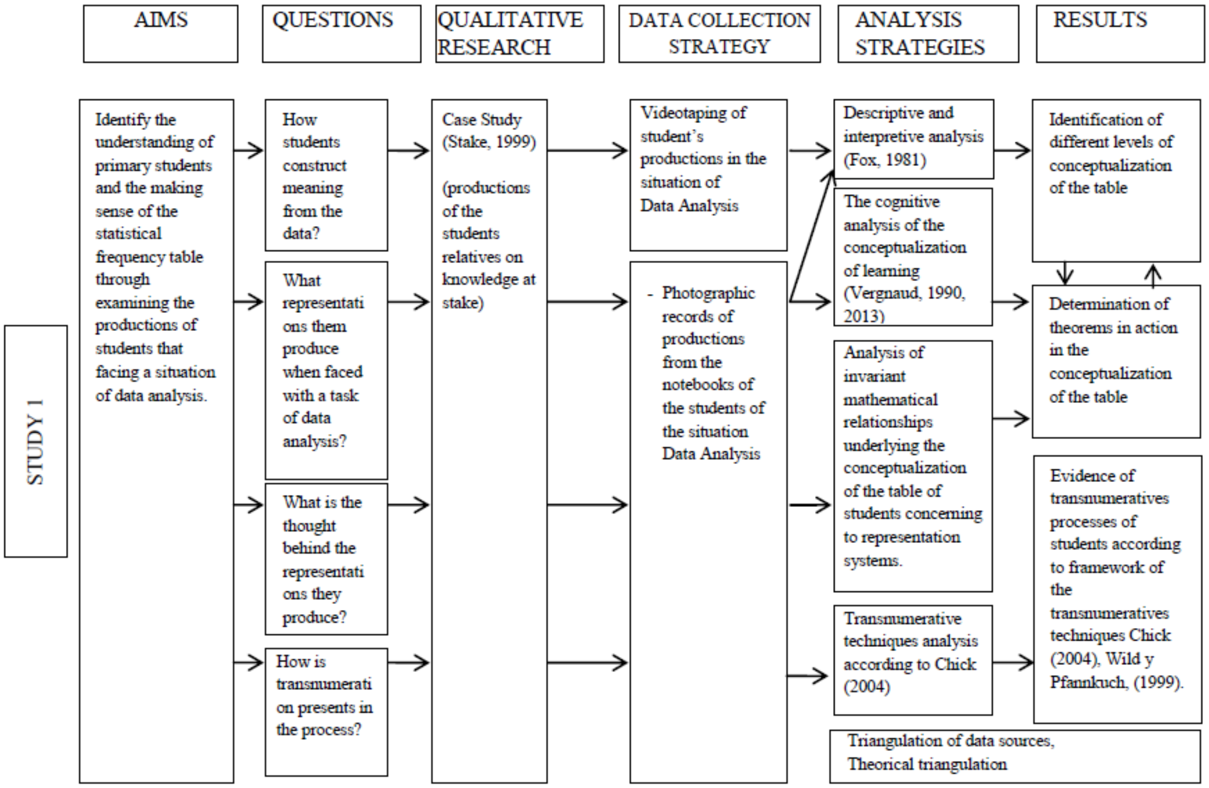


Figure 55. Methodological design of Study 1

2.2.1. Participants

Participants belonged to two third-grade primary school classes in the first semester of 2013 in urban schools chosen based on their accessibility to the professors. One class was made up of 38 students, 16 girls and 22 boys, ages 7 to 8 years. The other class had 42 students, 30 girls and 12 boys, whose ages ranged from 7 to 9 years.

2.2.2. Data Collection

A statistics learning situation was implemented with paper and pencil, looking for the emergence of tables; the data for this study emerges from the productions of the students faced with this situation. The situation is framed in the "Data and Probabilities" theme of the third-grade mathematics course of the Chilean curriculum (MINEDUC, 2012), which calls for students to, "Carry out surveys. Classify and organize the data obtained in tables and visualize it in bar graphs." The goal of the lesson that was implemented was to organize and classify data in order to obtain information. Based on this goal, a context was devised that would be interesting to students, related to the quality of the snacks they eat in school. Based on the snacks brought by the students, a printout was designed with this data displayed iconically (see Figure 57). The central question was, “How can we order and organize the data regarding our snacks to find out if we are at risk of contracting a disease?”

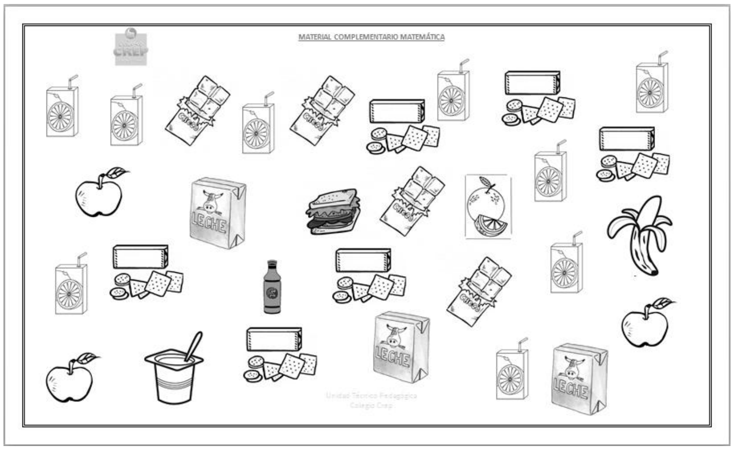


Figure 56. Handout that shows the preferred snacks brought to school by a grade 3 class.

Based on the premise that the data was real and would motivate the students, the teachers established the context of the snacks that the students bring to school every day. They asked each of the students before beginning the activity their ranking of the snack that they brought that day, and only one student responded “my favorite”. With the data collected, each professor created an image of all the snacks indicated by the students, and each one made a printout that represented the type of snacks and number of each type using icons. In the following lesson, after introducing the topic of nutrition, the professor gave a copy of the printout of the lesson's snacks to each student.

The situation proposed falls under a constructivist vision of learning, in which students are allowed to elaborate personal strategies using acquired knowledge and sharing ideas. For example, some initial interventions by students during lesson pointed out that the snacks were disorganized in the printout. The professor asks in return how to order them. The children begin to share ideas and the professor registers on the blackboard the possible strategies that the students name. Then, the professor comments, “Why did I ask you all about snacks? What could we find out in today’s activity?” (The students predict “learn more about diseases”, “eat healthier"). The professor affirms, "We'll see if today's activity can help you respond to that," and writes the challenge on the blackboard as a question.

How can we order and organize the data regarding our snacks to find out if we are at risk of contracting a disease?

2.2.3. Categories of Analysis

Given the importance of systems of representation in the process of conceptualization (Vergnaud, 1990) and the explicitation of concepts, we can expect that in the data analysis situation productions will appear based on systems of representation (Sureda & Otero, 2011). In resolving the snacks situation, we expect three systems of representation: numerical [NSR], referring to counting with numbers; iconic [ISR], referring to constructions with signs similar to the objects represented; and written [WSR], referring to written linguistic forms.

As the data analysis situation addresses scholastic statistics goals, it is important to recognize how the students move from one representation to another in order to better understand the data. To obtain evidence of the students' processes of transnumeration, Chick's (2004) framework of transnumerative techniques is used.

2.3. Results

The analysis of student solutions using TCF shows the relation between the students' conceptualizations, systems of representation (SR), and invariant operators (IO). This analysis allows us to recognize the process of conceptualization of the table, which begins with production of lists and becomes progressively more complex in various types of tables, until it reaches frequency tables.

The process detailed here shows that student productions in the same situation are guided by different invariant operators, that is, different schemes, sometimes lists, sometimes tables. This shows that, when knowledge of a conceptual field (CF) is incipient or not explicit, contradictory schemes coexist for the same concept. Even when the students only possess list-type schemes, they are used coherently to respond to the initial problem.

The table scheme is not unique, but rather there exists a grand variety of them, differentiated by the representations they use. As such, full command of the CF of tables should incorporate in its teaching the different systems of representation connected to the concept, which until now has been the result of implicit teaching.

The analysis of the results using TCF allows us to distinguish symbolic representations and deliver a partial explanation of their meanings, which give shape to the invariant operators used to face statistical situations that require organizing data. Constant variables were observed for dealing with the proposed situation, which is consistent with a previous epistemological study.

2.3.1. Production of statistical representations

Teachers propose an open-ended problem and give students freedom to choose solution strategies. They explicitly frame possible solutions using only the strategies proposed by students.

For the statistical data analysis situation and the corresponding student productions, the performance of each student was analyzed in terms of the possible schemes used, described according to their components[22], especially the invariant operators that may have been called forth in the interaction with the proposed situation and that account for the concepts and theorems in use. Later, the conceptualization processes shown by the students were categorized (in Appendix V.1 the protocol for this categorization is shown) based on the student productions, resulting in there being more than one production in the same category, evidence of the triangulation of the data source, according to Stake (1999:96).

[22]According to Vergnaud's concept *scheme*: *invariant operators* (concepts in action and theorems in action) that guide the subject's recognition of the situation's pertinent elements and the recollection of data in the situation to be dealt with; *anticipations* of the end to be achieved, of the expected effects, and the eventual intermediate stages; *rules of action* of the type "if... then" that allow for generalizing a series of actions by the subject; and *inferences* (or reasonings) that permit or obstruct calculating the rules and anticipations based on the information and the system of invariant operators available to the subject.

How do students carry out the task?

Here we cite some phrases that help to understand how the activity was carried out by the students. One of them drew a box and said, "*I'm going to make a box, and here I'm going to put what's healthy, and here what's not healthy.*" One student drew five columns and said, "*I'm ordering... It's a box, and I'm going to put fruit, yogurt, milk, and so on...* [pointing to each column]". One student asks permission to cut the worksheet and put each food item in order [he indicates in his notebook that he will place them in columns]. In other notebooks two-column tables are observed with headings chosen and written by the children "*healthy food"* and *"unhealthy food"*, *"bad"* and *"good"*, *"healthy"* and *"bad"*, *"healthy food"* and *"junk food"*, *"unhealthy"* and *"healthy"*. One student draws a frame around the foods on the worksheet that she considers *"unhealthy"*; another girl counts the food items and crosses them out on the worksheet in order to not recount them. Without counting, other students draw a representative image in horizontal lists; others draw the image inside tables with headings and cells. Some students draw as many images as appear on the worksheet [graphical tables like pictograms appear]. Some students create two-column tables with headings in which they draw the image and beside it the corresponding count. Other children create pictograms without separating in categories and without words. Only a few children create [frequency] tables using words and the count corresponding to each image and/or marginal totals.

2.3.2. Analysis of student productions

A characteristic of tables is that they contain all the data. Tables allow for communicating specific numerical ideas and are useful when individual values and the comparisons made with them are important (Estrella, 2014). Also, tables order and separate data.

Taking this into account, the student productions show -all with explicit categories-: tables with icons, with and without counting; tables with text, with and without counting; tables and/or lists with text and marginal totals; lists, with and without counting. Some of the productions lose information by not carrying out counts (lists; tables with icons without counting; tables with text without counting; tables with text and marginal totals) but resolve the proposed problem, and others contain the frequencies of the elements in each partition (tables with text and counting; tables with icons and counting; lists with text and counting) which allows for providing the response and placing all the data in the table.

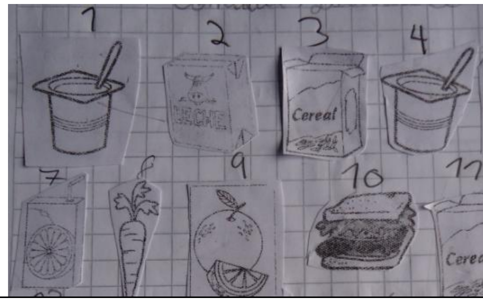
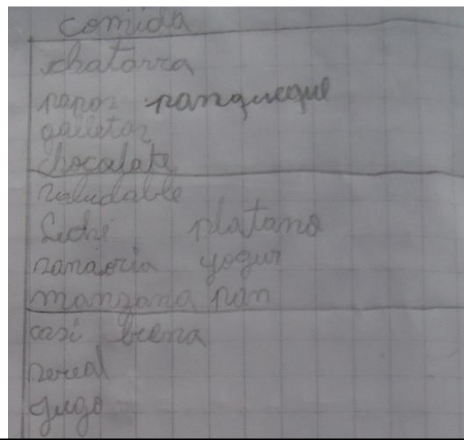
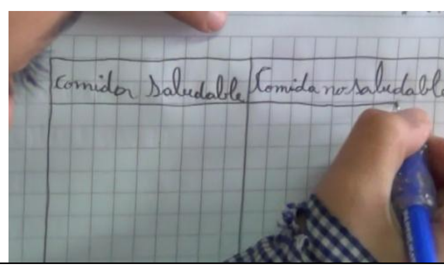
2.3.2.1. Transnumeration in student productions

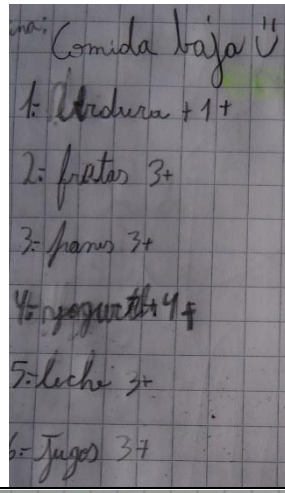
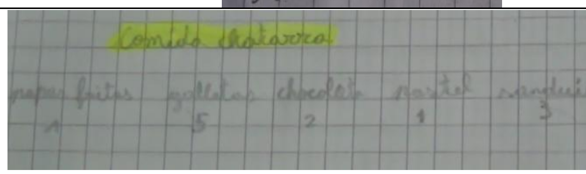
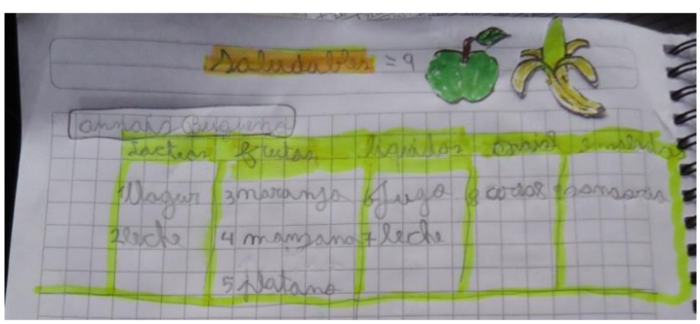
If we are interested in how students work with data to better understand it, then we are interested in a statistical education view of the question, "How does transnumeration appear in the process of conceptualization of a frequency table?"
The student productions showed evidence of transnumerative techniques (Chick, 2004) put into play in searching for and displaying the behavior of the data in the snack situation.

Of the ten techniques proposed by Chick, at least seven are observed in the lesson analyzed (they can be seen in the first column of Table 1, below). Among them: *ordering*, in which the data is organized according to a criteria (data is ordered according to how it appears on the worksheet); *grouping*, in which data is grouped according to a criteria, with the possibility of creating a new variable (the new variable “snacks’ nutritional quality” is created, for example, using snack data according to whether it is a healthy food, a somewhat healthy food, or an unhealthy food); *subset selection*, in which part of the data is selected for transnumeration (data associated with the quality of the snack, “good” and “bad” without considering points in between); *change of variable type*, in which a categorical variable is thought of in numerical or ordinal terms or vice versa (the type of snack -a categorical variable- can be shown in an ordinal manner, by ordering the snacks from most healthy to least healthy); *frequency calculation*, in which the frequencies of occurrence of values in each category of a categorical variable are calculated, creating a new variable (number of snacks according to their “nutritional quality”); *graphing*, in which some variables are graphed or tabulated (construction of graphs or tables of the number of snacks consumed according to their “nutritional quality”); and *other calculations*, such as the sum of each category for a variable.

Table 13 allows us to observe some of the transnumerations carried out by the students and their explanation.

Table 13
Examples of productions that show transnumeration techniques in the grade 3 snacks situation

Technique	Evidence	Description
Ordering		The data is ordered according to how it appeared and was cut out from the work sheet (for example, the yogurts are not grouped together, nor are the cereals).
Grouping		A new variable "snacks' nutritional quality" (implicit) is created using the snack data, with values "healthy", "middle" (almost good), and "unhealthy" (junk food).
Subset selection		Data associated with the snacks' nutritional quality is considered only as "healthy" and "unhealthy" ("middle" or "almost healthy" is not considered).

Change of variable type		"Snacks' nutritional quality" (categorical variable) can appear in an ordinal manner, by ordering the snacks according to their degree of healthiness (use of + sign).
Frequency calculation		Number of types of snacks in each subcategory of the variable "snacks' nutritional quality".
Graphing		Construction of a frequency graph of the nutritional quality of the "healthy" snacks.
Other calculations		Sum total of the frequencies of each category for the variable "snacks' nutritional quality": healthy and unhealthy.

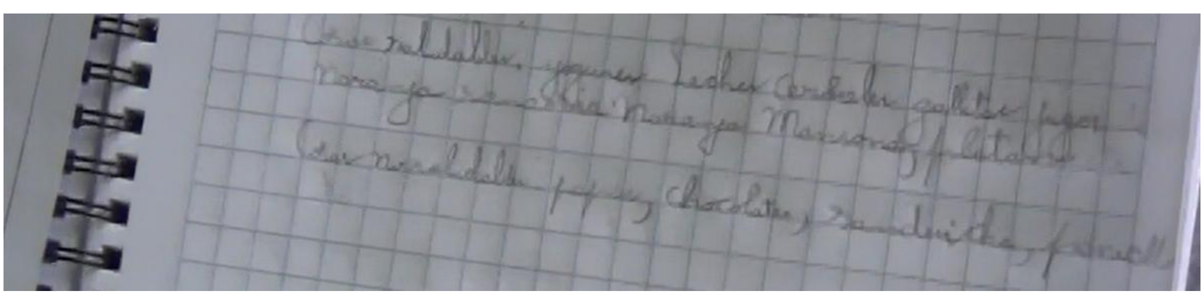
2.3.2.2. Analysis of conceptualization according to systems of representation

The conceptualization observed in the student productions shows three distinguishable stages: lists, pseudo tables, and tables. Each stage possesses its own characteristics: (1) The list stage (lists with icons without counting; lists with text without counting; lists with icons (with or without ordered repetition) and counting; lists with text and counting, see summary of these productions in Appendix V.2); (2) The pseudo table stage (tables with icons without counting; tables with text without counting; tables with icons and counting; tables with text and counting), and (3) The table stage (tables with icons without counting; tables with text without counting, tables with icons and counting; tables with text and counting).

Next, we analyze the systems of representation: numerical [NSR], referring to counting with numbers; iconic [ISR], referring to the construction of similar signs, and written [WSR], referring to written linguistic forms. Next, we present and describe the solutions of five students that are representative of the three stages of conceptualization that we found in the student productions for grade 3 for the snacks situation according to the theorems in action that the students presumably use, and also identify the possible transnumerative processes (Chick, 2004).

Production 1. List with text without counting.

The production shown here shows a list for the variable (types of snacks) and its categories (healthy and unhealthy), with data represented by text without counting. This type of production is one of the most basic, as in presenting a sequence of words without counting or repetition it loses crucial elements for responding to the situation, even when a sort of subitizing or numerosity can be perceived regarding a larger set versus a smaller set of written elements. As such, an implicit NSR [iNSR] is assumed, which allows for assuming that one set is a subset of another larger set.



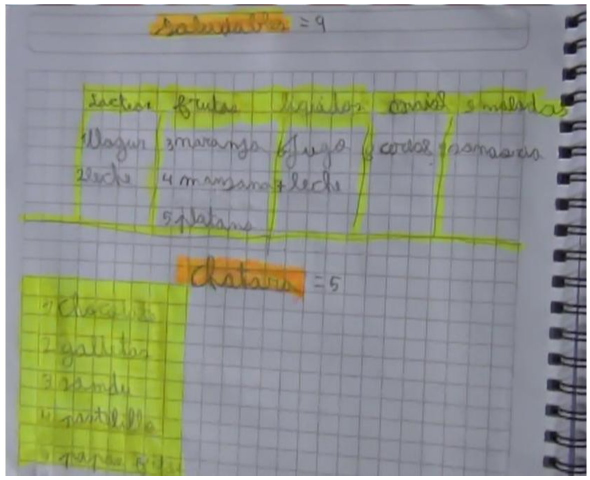
In the following table, the theorems in action that appear to guide the student's actions in each system of representation are described:

ISR	WSR	iNSR
Theorem: "Each icon (snack) can be represented by a word in written language"	Theorem: "An equivalence relation determines a partition."	Theorem: "For each pair of sets, one of these is a subset of the union of both."
Theorem: "A collection is determined by listing its elements by extension."	Theorem: "Something belongs or does not belong to a class (belonging relation)."	Theorem: "If every element is an element of A if and only if it is also an element of B, then A and B coincide."
List		

The student that places a representative for each lesson passes from the given set to the quotient set using the theorem "an equivalence relation determines a partition". The phrase "*can be represented*" in the previous table indicates that there can be more than one representation, but the chosen representation should be maintained: if the word *fruit* is chosen for the apple, then each time that the apple icon appears, the same word must be used.
Of the transnumerative techniques, only ordering and grouping are observed.

Production 2. Pseudo table with text without counting.

The following production corresponds to a pseudo table for the variable with its categories and sub-categories, with data represented using text with the ordinal number of each element of the category and the cardinal number of each category. In this type of ordered production with marginal totals, the table created collects information, but it loses part of it by not considering the cardinality of each element of each sub-category. The student displays a category with sub-categories in a table with a title, headings, rows, columns, and cells without counting (frequency). It presents the other "simpler" category in a format similar to a vertical list, which could be an indicator of the intermediate stage of its tabular scheme, which moves between list and table.

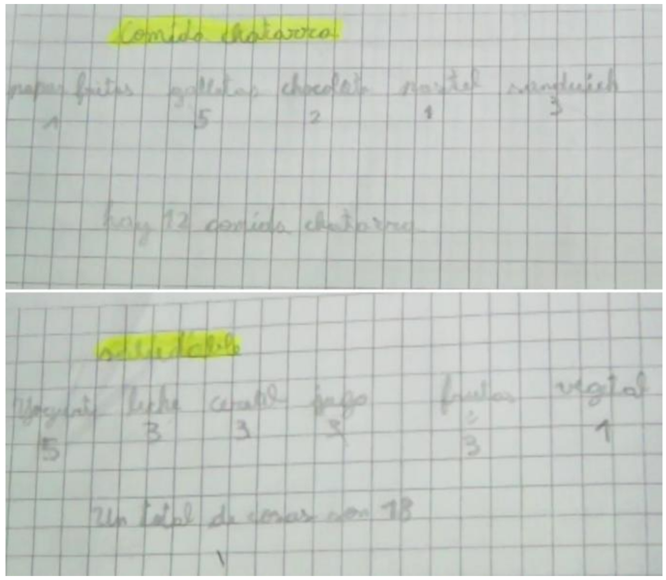


ISR	WSR	NSR
Theorem: “Sub-classes can be separated using segments between them.”	Theorem: “An equivalence relation determines a partition and a partition defines an equivalence relation.”	Theorem: N is a totally ordered set. (Given that there exists an ordering relation in N, then the cardinality of the set "healthy" is greater than the cardinality of the set "unhealthy")
Theorem: “A collection is determined by listing its elements by extension.”	Theorem: “Headings let us name sub-categories within a partition.” Theorem: “If A is a subset of B, every element of A is part of B."	Theorem: “All the ordinal numbers make up a class; every finite well-ordered set is orderly isomorphic to exactly one natural number."
Pseudo Frequency Table		

Among the transnumerative techniques that come forth in this production, there is evidence for grouping, change of variable (among healthy snacks new sub-categories come forward, for example, fruits, dairy, liquids), graphing/tabulation, and other calculations (sums). Although this production has dividing segments and a certain spatial order, because there is no frequency calculation, the associated tabulation is not found either. Also, some sub-categories are not disjoint (dairy and liquids) and lead to other calculations, such as the category total, giving erroneous results.

Production 3. Table with text and counting.

The following production shows a table with variable categories, although lines are not drawn, with data represented by text, with counts for each class and marginal totals for all the classes of the category. The absence of delimiting lines is not an impediment to the fluid reading of the data representation used, as the spatial order and alignment of text and numbers allows us to infer the use of a title, headings, rows, columns, and cells.

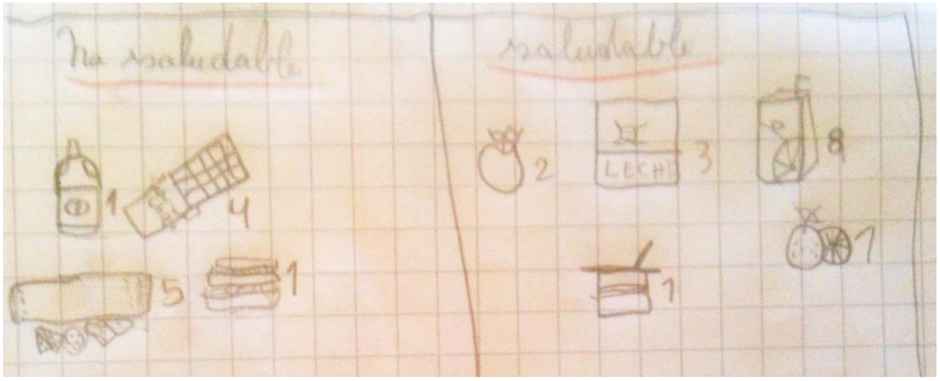


ISR	WSR	NSR
Theorem: “Each icon can be represented by a word in written language”	Theorem: “An equivalence relation determines a partition and a partition defines an equivalence relation.”	Theorem: N is a totally ordered set. Given that there exists an ordering relation in N, then the cardinality of the set A is greater than the cardinality of the set B.
Theorem: A collection is determined by listing its elements by extension.	Theorem: Something belongs or does not belong to a class (belonging relation).	Theorem: Card (A∪B) = Card (A) + Card (B) – Card (A∩B)
Frequency Table		

The transnumerative techniques observed are grouping, frequency calculation, tabulation, and other calculations (category sum total).

Production 4. Pseudo frequency table with icons and counting.

The snacks learning situation uses a worksheet with images, and as such, many students used the icons that represent these snacks, either drawn or cut-out from the worksheet. The following production presents a pseudo table with the variable categories, with data represented by icons with counting, and with a line drawn to separate the categories. Although there is a count for each category class, a line delimiting the categories, and an icon that represents the heading, it is not considered a frequency table because rows, columns, and cells are not defined to give a spatial order and allow for a sequential vertical or horizontal reading.

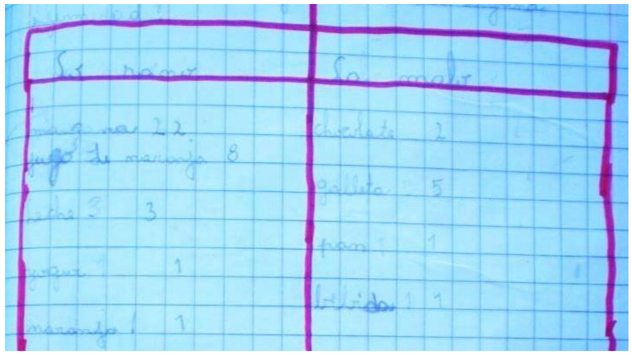


ISR	WSR	NSR
Theorem: "Categories can be separated using segments between them."	Theorem: An equivalence relation determines a partition and a partition defines an equivalence relation.	Theorem: Card (A∪Ø) = Card (A)
Theorem: each icon is a class in the category.	Theorem: Something belongs or does not belong to a class (belonging relation).	Theorem: The cardinality of a class is the sum of the cardinalities of each sub-class.
Pseudo Frequency Table		

Among the transnumerative techniques brought into play, we find grouping and frequency calculation.

Production 5. Table with text and counting.

This table presents the headings of each column that correspond to the name of each category for the variable, and the body of data includes the name of the class and its corresponding frequency in each row. There is no demarcation between the class (nominal qualitative aspect) and its frequency (quantitative aspect), but it appears that the student, in making the table, carries out a segmentation, as an effort to make an evident physical separation between text and number is observed.



ISR	WSR	NSR
Theorem: Each icon can be represented by a word in written language.	Theorem: An equivalence relation determines a partition and a partition defines an equivalence relation.	Theorem: Card (A∪Ø) = Card (A)
Theorem: A collection is determined by listing its elements by extension.	Theorem: Something belongs or does not belong to a class (belonging relation).	Theorem: The cardinality of a class is the sum of the cardinalities of each sub-class.
Frequency Table		

The transnumeration techniques observed are grouping, frequency calculation, and tabulation, the latter in its habitual form of horizontal reading of each class and its associated count.

Based on the student productions, we deduce that all tables can be associated with a partition, but not all partitions are associated with a table. The children that presented a more complete concept of tables performed better in responding to the problem than those students with a less complete concept of tables. Only 14 subjects in a class of 42 students and 3 in a class of 38 delivered a numeric response to the situation proposed. The rest of the students either did not give this type of response or used the perception of numerosity through subitizing or estimation to determine the greater or lesser cardinality of the set.

3. Discussion and Conclusions of Study 1

Considering that the theory of conceptual fields is based on the principle of pragmatic elaboration of knowledge, this study lets us approach the answers to questions such as: How does the notion of tables emerge in students in the first years of school? How do students construct meaning from data? What is the thinking behind the representations that students produce? What representations do students produce when faced with a new data analysis task? What levels of conceptualization are reflected in these representations?

Discussion regarding transnumeration

The process of socialization of the meanings of tables through collective sign systems begins before students enter the education system, and, as such, it is natural to assume that the participants in this study, who have at least three years of schooling, recognize the word “table”, although with different meanings. The students’ performance in resolving the situation was diverse, due possibly to their previous dominance of some semiotic representations and similar situations which they had faced previously. This means that the meanings already known for tables could have formed a foundation for learning, but also an obstacle [an epistemological obstacle or a didactic obstacle, per Brousseau (1987)], and, as such, the meaning of the table object requires explanation in order to be socialized and consolidated according to the sign system that children learn in mathematics of science classes.

Tables are not only used to register data, but also to look for and compare data, to find and display relationships in data. Exploratory data analysis seeks to get to know the behavior of the data, and the process of transnumerating the data is activated to obtain greater understanding. In the student productions, it is possible to confirm the four phases of transnumeration identified in the literature: recognition of the data’s message (snacks can be healthy or not), choice of representation (types of lists or tables), transformation of data (from icons to text, numerical frequencies), and representation of transformed data (production of iconic lists or text lists, iconic tables or tables with and without frequency), Rubick (2004). In these phases, the transnumerative techniques that specifically appear in the student productions are grouping, sub-set selection, change of variable type, frequency calculation, graphing/tabulation, and other calculations.

Questions referred to “How do students create meaning from data?” and “What representations do students produce when faced with a data analysis task?” are analyzed using transnumerative techniques, as the schemes that are called forth are not only mathematical, but also statistical, in which data and context are relevant, and representations are necessary for communicating the findings. In order to approach the answers to these questions, the process of transnumeration gives us tools that reveal the meaning of organizing data to obtain information. For the learner, the construction of meaning from data is relevant if the categorization is adequate and disjoint, and the counting of the elements in each class and category allows the learner to respond to the question of the initial problem. The productions turned out to be homogeneous in different groups, in different institutions and with different times and teachers (triangulation of data sources), which allows us to suggest that the representations produced by third grade learners faced with a data analysis task correspond to the stages of lists, pseudo tables, and tables that we have described.

In the transnumerative analysis carried out, a frequency table must show at least the use of the techniques of grouping (where the disjoint categories of a variable are present), frequency calculation (counting of the respective class), and tabulation (which allows at least for a localization in a row and a column of the element of the class and/or its respective count).
In using more than one theoretical scheme in the interpretation of the phenomenon under investigation, we have recurred to triangulation among theories, understood as distinct views from which the phenomenon can be interpreted (Denzin, 1997:319).

Discussion regarding conceptualization

Tables are an object that exists in all cultures and in all time periods, and as a symbol system they are mediators of reasoning (Estrella and Mena, 2013). In their role as statistically literate citizens, students need to be competent in dealing with statistical representations. As in Brizuela and Alvarado (2010), this study assumes that mathematical work, and in particular statistical work, can be increased if children have access to notation tools -like tables- that are part of mathematics and science classes. Also, we assume that the task of constructing representations promotes greater cognitive interaction than using ready-made representations (Cox, 1999).

The empirical data from the study allows us to characterize a progressive trend in the conceptualization of frequency tables, which includes elements from iconic list schemes and text lists without counting to tables with counting and frequency tables with marginal totals. However, it is not possible to affirm that the conceptualization process of each student passes through each of the stages described, although it is probable that the previous mastery of data organization situations has an impact on the progress, set backs, and recovery of knowledge that are produced in the process of conceptualization of tables.

This study allows us to observe some special characteristics in the student productions. For example, in production 3, the student did not draw lines in the table, and in agreement with Sugio, Shimojima, and Katagiri (2012), the subject was able to mentally segment and organize the data in rows and/or columns in a particular direction, which allows tables to be read efficiently. In the terms of Trouche (2009), an artifact arranges the subject, relatively pre-structures his or her activity, and influences the schemes that he or she will construct.

A recurrent factor in these student productions, in the three stages of conceptualization of the table proposed, is repetition, with or without order, of the elements of a class; this makes us suppose that ordering the repeated elements would facilitate quantitative visual estimation and, as a consequence, also facilitate counting and efficiency in visual search processes.

The cognitive framework adopted lets us describe not only the conceptualization of frequency tables and the limits of this conceptualization, but also explain and predict other conceptualizations. The categorization proposed by Vergnaud takes into account the mathematical structures underlying the situations and the psychological development of the conceptions for dealing with these situations. It is possible to infer that the mathematical concepts used include principally: classification; addition; counting (bijection and cardinality); the commutative and associative properties; existence of the identity element; comparison of numbers (meaning of number); comparison of graphical elements and/or comparison of two cardinals; definition of set by comprehension and by extension; finite set; union of sets; a set as the complement of another set; cardinal and ordinal. Considering the discrete character of frequency tables, the processes activated are those of association, differentiation, and perception of order and quantity. The theorems in use in all the productions were (1) "an equivalence relation determines a partition and a partition defines an equivalence relation," related to the processes of association and differentiation; and (2) “$\text{Card}(A \cup B) = \text{Card}(A) + \text{Card}(B) - \text{Card}(A \cap B)$,” concerning order and quantity.

The conceptualization of tables in these students shows different inherent representations, such as text lists without counting, text lists with counting, tables with icons with and without counting, tables with text with and without counting, and tables with text without individual counts but with marginal totals. This conceptualization of tables in this group of students allows us to shed light on a group of situations: referent, (for example, situations of registering data in a table, of counting in a table, of listing elements belonging to a class); a set of invariant operators, signified, “for example, partition, equivalence relations, and counting that allow for ordering data to obtain information); and a set of representation, signifier, (for example, physical segmentations, rows, columns, cells, headings, written language).

Although in the conceptualization of tables different representations were identified, in observing the productions of the two groups of students from different schools, we observe that the organization of the activity does not vary, as regularities were identified among productions of the same grade level, in the way they address and develop the same situation.

The table as an external symbol is visually processed through the recognition of its properties and spatial relations, and as such is symbolically recognizable. The analysis carried out leads us to hypothesize that children could progressively develop different schemas for the table concept, depending on the learning experiences they have encountered initially, for example, the construction of iconic lists without counting and with evident repetition of elements and spot and sequential readings associated with these lists. This type of experience would allow them to appreciate the advantages of frequency tables, with or without demarcating lines, in which the differentiation and order is evident, which activates processes of searching or comparing through sequential readings (in rows and columns).

The majority of current mathematics curricula include a theme of statistics in which the use of tables is explicit from the earliest levels. Identifying transnumeration techniques in the process of conceptualization of tables is useful for understanding the foundations of the development of statistical thinking that statistical literacy creates by including basic abilities used to understand statistical information, such as the ability to organize data, create and present tables, and work with different data representations.

Empirical study delivers insights for proposing the curricular introduction of tables after lists and to show textbook developers and teachers that students’ conceptualization of tables can be found in different stages in the same classroom, even different from the understanding of students who may have explicitly learned frequency tables.

The results of the observed productions, which allow us to analyze the intrinsic knowledge of the students in their statistical performance, allow us to suggest a progression in conceptualization that goes from lists to pseudo tables to tables. Based on this progression, we suggest:

(1) Prior to conceptualization of tables, consider activities that consolidate the capacities of association and differentiation, ordering, and quantity;

(2) To teach the table concept, first introduce situations with the list concept -which is the basic unit of tables- and its spatial dispositions that permit vertical or horizontal readings;

(3) Make explicit and value the communicative components (headings with names of the variable, its categories, and classes);

(4) Consider various situations that provoke the necessity of creating tables, allowing for different systems of representation (iconic, written, and numerical); and

(5) In this conceptualization, allow for creating tables with and without physical demarcations, as long as the mental segmentations facilitate spot readings (cells), sequential readings (rows and/or columns), and/or global readings (the entire table or parts of the table).

The principal goal of TCF is to identify the principal continuities and ruptures in knowledge that finally become concrete in the teacher's mediating actions. It is the teacher who will evaluate the opportunity for utilizing this continuity and encouraging the student to move from one class of situations to another, similar to the first, and facilitate the student making this step without difficulty. Or, the teacher may also consider it appropriate to utilize a rupture, as a way of provoking an imbalance between the situation to be dealt with and the students' capacities, and make them aware of the limits of their current points of view.

Recognizing the theorems being used that support the process of progressive conceptualization of tables provides us with knowledge to create learning situations that consider continuities and allow for confronting ruptures, to be able to educate students competent in the comprehensive use of the tabular format. Recognizing the transnumerative techniques allows us to educate statistically literate students who use representations to look for better understanding of data.

Tables should receive explicit teaching in diverse manners, in their form and content, so that they can truly offer children the opportunity to organize the data with which they interact in a meaningful way. These findings help to make sense of the tables that the participants create (understanding the sense of a concept for a subject as its relation to situations and to the signifiers of this concept, that is, the schemes called forth by the subject in these situations or their signifiers) and go deeper into the ability to understand data and competence when faced with tabular representations.

STUDY 2: COGNITIVE DEMANDS IN A PRIMARY SCHOOL DATA ANALYSIS LESSON

1. INTRODUCTION

In this study a statistics lesson is described -two implementations of the same lesson- and the changes that occurred in the planning of this lesson, which shows part of the process of how classes were continually developed in a lesson study group[23]. We used Garfield and Ben-Zvi's (2008) statistical reasoning learning environment (SRLE) and analysis of tasks and factors associated with maintaining high level cognitive demands or a decline in the level of cognitive demand (Stein & Smith, 1998; Stein et al., 2000). The research strategy was a case study, and the goal was to describe and interpret the change in a lesson implemented from the perspective of task analysis and to present a snapshot of how one Chilean teacher's understanding gradually improves her classes based on a learning environment that promotes statistical reasoning, supported by the activities and reflections of the lesson study group.

The study hopes to respond to questions of a didactic nature:

What characteristics does a teaching task aimed at data analysis have?

How does the teacher organize a data analysis lesson in primary school?

How does the teacher maintain the task's level of cognitive demand?

2. ANALYSIS OF THE LESSON USING STEIN AND SMITH'S COGNITIVE DEMANDS.

2.1. Introduction

The motivation for working with statistical topics is due to many professors having limited ideas (often erroneous) about basic concepts in this discipline (Del Pino & Estrella, 2012), and also because it is a new topic that the Chilean school curriculum includes from the first years

[23] A group of teachers that met periodically (three hours per week during two months) to prepare and study lessons for teaching curriculum content. The group meetings were held by three researchers, one of them is the author, who was responsible for the situation referred to as "Snacks".

of school, issues that highlight the importance of teachers having knowledge of the principles and concepts related to data analysis.

To approach the answers to the questions that guide this study, a data analysis lesson[24] is planned whose goal is “to classify and organize data in order to represent it and obtain information”, planned in a group of teachers who study the lesson according to the Statistical Reasoning Learning Environment (SRLE), which aims to develop in students a deep understanding and development of statistical reasoning, combining materials, activities, classroom norms, support, and discussion.

This study assumes that the tasks used in the classroom determine the type of learning created by students. The concept of cognitive demand defined by Stein et al. (2000) is considered, as well as the kinds of cognitive processes involved in solving a mathematical problem, in its first phase of understanding the task, as well as in carrying out the task; processes that begin with memorization and the use of simple procedures and algorithms -a low level cognitive demand-, and reach the use of complex strategies of thinking and reasoning characteristic of mathematical thinking -a high level cognitive demand-.

2.2. Methodology

This study adopts a qualitative focus and details a case study to identify the level of cognitive demands that a teacher evokes in managing a data analysis lesson. The methodological design is shown in figure 58

[24] A lesson already described in Study 1.

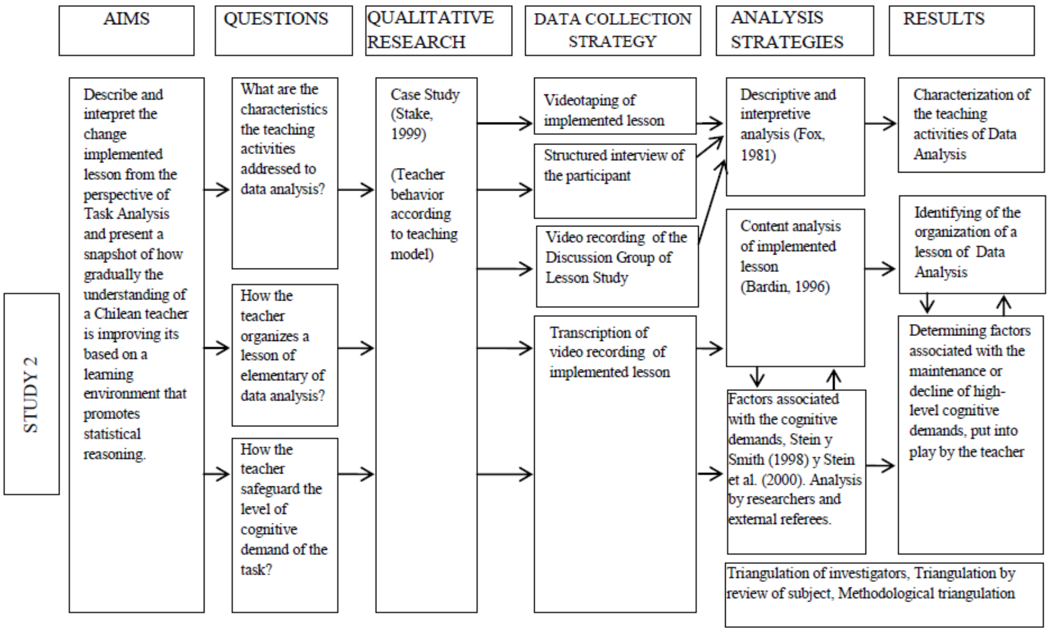


Figure 57. Methodological design of Study 2

2.2.1. Participants

The school and the teacher

The school is a private establishment subsidized by the city of Quilpue, Valparaiso, Chile. The student body ranges from preschool to the end of middle school, and is recognized as belong to the group of “middle" socioeconomic status. Its performance in national mathematics exams is lower than the national average (-31 points). The course in which the lesson is implemented is grade 3, included in the establishment’s primary school, which reaches grade 5.

The teacher, who we will call Teacher C, has a primary school teaching degree, has five years of experience, and teaches mathematics classes in grades 1, 2, 3, and 4. She is part of the group of ten teachers who were selected to participate in a statistical education project[25] of the Pontifical Catholic University of Valparaiso and the Center for Advanced Research in Education; she is a participant characterized by her enthusiasm and proactive attitude in the lesson study group convoked by the project.

[25] CIAE Project 05 2012-2013 “Impact of public classes in basic education teachers’ beliefs and of lesson studies on the quality of statistics teaching" by researchers Raimundo Olfos and Arturo Mena-Lorca.

Lesson preparation

The lesson was held in the second week of May, 2013 by the teacher. The class is grade three, with 38 students, 16 girls and 22 boys, ages 7 to 8 years.
Teacher C belongs to the group of five teachers who make up the lesson study group for grade 3. Together with the researcher, five teachers designed a teaching situation for the data and probabilities curriculum content. The researcher proposed to the group of teachers that they work using the SRLE model of statistics teaching -although without its technological component-, to teach the fundamental ideas of statistics and develop classes with a high cognitive demand, issues addressed in each weekly meeting (a total of 10) of the lesson study group.

The teaching design focused on the development of understanding of two fundamental statistical ideas: data and data representation. The directives of the SRLE model include the use of data that is real and interesting to the students, collaborative class activities based on investigation, with work norms that include statistical discourse and argumentation centered on statistical ideas and an alternative evaluation to understand what students know and how they develop their statistical learning, integrating the evaluation of the plan and the teaching progress.

As described in Study 1, the learning situation addressed the theme of data and probability, which asks grade 3 students to "carry out surveys, classify and organize the data obtained in tables and visualize it in bar graphs." The goal of the lesson was to organize and classify data in order to obtain information. Based on this goal, the professors devised a context that would be interesting to students, related to the quality of the snacks they eat. The central question was, “How can we order and organize the data regarding our snacks to find out if we are at risk of contracting disease?”

The teacher later gave each student a handout (see figure 57 in Study 1); in the lesson plan elaborated by the lesson study group, the teachers specified the student activity, for example, “Observe your worksheet about snacks that were mentioned in the last lesson and discuss with your partner how you would make a classification" (see lesson plan in Appendix V.3).

2.2.2. Data Collection

Figure 59 shows the data collection design regarding the lesson design and its implementation, related elements that allow for determining the factors associated with the level of cognitive demands that the teacher manages.

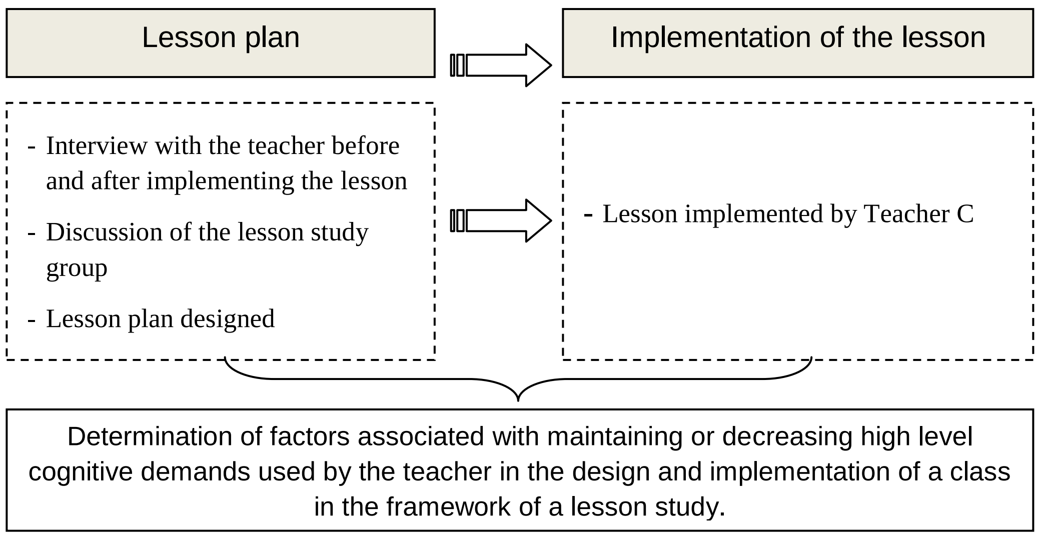


Figure 58. Collection of data in the design and implementation of the class.

Having considered different ways of addressing the phenomenon -filming, individual interviews, and discussion groups- gives us methodological triangulation by involving the use of different methods to observe the phenomenon. Additionally, triangulation of subject verification was implemented, in which the teacher being researched examines and confirms or disconfirms what was written about her and her lesson (Denzin, 1997).

The lesson

The filming and the examination of the images are textually transcribed and complemented by images of the students' notebooks that show the strategies used in the mathematical classroom activity. In this study we analyze a complete filmed lesson, specifying the tasks proposed by the teacher and the tasks actually carried out by the students.

The lesson carried out corresponds to the second implementation of the lesson, of a total of three, studied by this lesson study group. The first lesson was implemented by another teacher from the same group, with Teacher C attending as an observer. For reporting in this work, the second lesson was chosen -carried out by Teacher C- in order to present the improvement in the level of cognitive demands placed on the students with respect to the first implementation of the lesson (the third lesson was characterized by largely presenting maintenance of cognitive demands, and as such, between this and the first lesson, which was characterized by decline in the cognitive demands, the second lesson represented the most typical situation).

Interviews with the teacher

As an additional data source, filmed interviews were carried out and later transcribed with Teacher C before and after implementing the lesson, regarding her opinion on the lesson and the discussion of the lesson in the lesson study group. The interview dealt with what the teacher considered important in the lesson and her evaluation of the discussion in the lesson study group. The data that the interview contributes is essentially a better understanding of how the teacher experienced what happened in the classroom and the prior work of planning the lesson.

Lesson study group Discussion

Another data source was the discussions from the teachers' group, filmed and transcribed. These discussions contributed to understanding how the teacher and the group of teachers discussed the design, implementation, and improvements of the lesson. Before implementing the lesson -before the lesson given by Teacher C-, the teachers of the lesson study group discussed the lesson they had created. Also, after the lesson was implemented, the group learned about the teacher's evaluations of the lesson and the teachers were able to see and discuss some episodes from the lesson implemented by the teacher -key episodes in the filmed lesson chosen by the investigating professor-.

2.3. Results

We analyze the text that registers the activity developed in the lesson. The lesson is fragmented in episodes that correspond to parts of dialogues with a complete meaning; for example, when the teacher states a challenge to which the students respond with explanations or questions related to the issue and the corresponding feedback from the teacher. The data analysis was carried out through a detailed interpretive description (Fox, 1981) and through content analysis (Bardin, 1996). The first analysis lets us organize and understand the knowledge stated by the teacher in the design and implementation of a statistics data analysis lesson, and the content analysis of each defined episode lets us identify fragments of information related to the factors of maintaining or decreasing the cognitive demand (see table 14).

The focus of the study is cognitive demand, and as such, content analysis is used as the principal tool for obtaining results related to teacher management of cognitive demands in the implementation of the lesson. The system of categories used is based in Stein and Smith's conceptualization of factors associated with maintaining or decreasing high level cognitive demands. The categories of analysis are codified on top of the text of the transcription of the lesson. Initially, two experts (the minimum for coders indicated by Krippendorff (1990)) discuss the coding that was done and agree on a system of categories with an inter-coder confidence of 99% according to the scale[26] proposed by Fox (1981); finally, the coding is externally validated by two more coding judges (using the same categories as the transcribed lesson). Regarding the validity of the study, we base our reasoning in the fact that the category system was previously formulated with characteristics of mutual exclusivity, homogeneity, pertinence, objectivity, fidelity, and productivity (Bardin, 1996:92).

The Task Analysis Guide (see Table 1 in Chapter II) was used to define the cognitive level of the tasks proposed by the teacher and the tasks carried out by the students. *Memorization tasks* and *procedures without connections to the tasks* were considered low level cognitive demands, while *procedures with connections to the tasks* and *doing mathematical work (statistics)* were considered high level demands.

The factors associated with maintaining or decreasing high level cognitive demands allowed us to recognize the principal factors that influenced the implementation of the tasks by the teacher, and these correspond to the category system on which the content analysis is based. In Table 14, seven factors associated with maintaining high level cognitive demands and six common factors associated with decreasing high level cognitive demands are described.

[26] Percentage agreement = 100 x number of data units coded equally / number of data units coded

Table 14

Factors associated with maintaining or decreasing high level cognitive demands (Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2000)

Codes for factors of maintaining or decreasing	Explanations
	Factors associated with maintaining high level cognitive demands
M1	Structure is provided for student thinking and reasoning.
M2	Students are given the means to monitor their own progress.
M3	Teacher or capable students give an example of high level performance.
M4	Teacher pressures students to give justifications, explanations, and meanings through questions, comments, and feedback.
M5	Tasks are based on students' previous knowledge.
M6	Teacher makes frequent conceptual connections.
M7	Sufficient time is given for exploration; neither too little nor too much.
	Factors associated with decreasing high level cognitive demands
D1	Problematic aspects of the task become routine.
D2	Teacher adjusts the emphasis of meaning, concepts, or understanding of the exactness or complexity of the answer.
D3	Insufficient time is given for dealing with the most challenging aspects of the task.
D4	Classroom management problems impede a sustained commitment to high cognitive level activities.
D5	The task is not appropriate for the student group.
D6	Students do not have to account for high level products or processes.

As pointed out, the teacher’s performance was evaluated regarding maintaining or decreasing cognitive demands; the coding was carried out by the author and an expert for the first and second implementations of the lesson. Both codings followed the same process. They were carried out based on the lesson transcriptions, the choice of factors for the episodes was discussed and agreed upon by both experts (inter-coder agreement, per Krippendorff (1990)), and to settle any differences the specific filmed episode was watched; that is, the same phenomenon was observed and coded with the same categories by different experts, which is a researcher triangulation index (Denzin, 1997). Later, another similar triangulation was carried out to obtain independent observation points. Two external judges were solicited to independently code the transcription of the filmed lesson with the same rubric of factors. The comparison of the functioning of the coding-categorization system allowed us to corroborate the findings of maintaining or decreasing cognitive demands put into play by the teacher and confirm the reliability[27] of the system (Appendices V.4 and V.5).

The following shows the lesson structure and its principal task together with the interpretative descriptive analysis. Finally, we summarize how the teacher maintained the level of cognitive demands in the task in the lesson implemented with content analysis.

2.3.1. Lesson structure

The lesson was planned for 90 minutes, but really took about 80 minutes. The activity's varied contents, the teacher's interventions, and their durations are described succinctly in the following sequence:

1. The teacher provides motivation for the issue by reading a letter from a child in situation of obesity (7 minutes, 2 seconds). The teacher activates the students' role through questions about the children's previous experiences regarding health and food, [lines 3-22].
2. The teacher asks the students to remember what she asked them. While she passes out the handout with the images, the teacher continues stimulating the students' interest with the snacks survey carried out previously. A student states that this image represents their responses about snacks, and the students affirm that the drawings are their snacks (2 minutes, 54 seconds). The teacher gives the students time to observe the image [lines 23-39].
3. The teacher asks, "Does anything catch your attention about the snacks?" A student points out that they are their snacks. Another student observes that the snacks are disordered. The teacher asks, "What could we do with this image that looks disordered?" (1 minute, 31 seconds), [lines 40-46]
4. A student points out that they could order the images. The teacher asks in return, "How could we do it? Does anyone remember how we can order things?" The children begin to give ideas and the teacher registers the majority of the contributions in a column on the chalkboard: *"order from smallest to biggest", "order in rows", "order by healthy and*

[27] To help in the reliability or coding agreement that the judges obtain in applying the category system, an instruction sheet was created for the coder, in which in addition to introducing the system (Stein and Smith's factors) with the definition of the categories used, examples were given of coded text.

unhealthy", "color the unhealthy snacks red and mark them with another image", "regroup", "tables" (2 minutes, 19 seconds), [lines 47-61]

5. The teacher presents the problem: Why do you think that I am asking you about snacks? A student responds, *"Because of the story* [the letter]." What could we learn with today's activity? *"About diseases", "Health".* The teacher proposes the lesson challenge and writes it in the form of a question on the blackboard. How can we order and organize the data about our snacks to find out if we are at risk of contracting one of the diseases mentioned? The students ask how to do it, and the teacher responds several times, "However you all want to," and establishes as the lesson norm that they can share their strategies and work and comment with their partner, (8 minutes, 3 seconds), [lines 71-100].
6. The students work freely and independently on various proposals to solve the challenge, (33 minutes, 29 seconds).
7. The teacher continually monitors the students' varied strategies and chooses three of them to write their work on the blackboard and show the class (7 minutes, 31 seconds).
8. The teacher asks the students who have not come to the blackboard: Do you remember why we are doing this? How are we eating? How is the majority of the class? What will happen if we continue this way? (3 minute, 35 seconds), [lines 159-170]
9. The teacher invites the students to compare their work with the work presented on the blackboard. What is the same? What is different? Is what we are doing related to mathematics? Students respond "*Tables!... Order!... Numbers!..."* (4 minutes, 34 seconds), [lines 176-194].
10. The teacher proposes choosing one of the three strategies presented by the students, "Which do you all think is the best option for responding to our challenge?" (Almost all the students propose choosing the option that corresponds to a frequency table). Could we find out if we are at risk of contracting a disease? *"Yes!"* I ask them, "What have we learned today?" *"To eat healthy and avoid some foods that are bad for us."* The teacher summarizes, indicating without giving it a name, "This is the clearest [table] for responding to the question," (5 minutes, 35 seconds), [lines 196-209]
11. The teacher gives them time to copy the table strategy in their notebooks. A student notices an error in the table's counting and tells the teacher. The teacher mentions the error to the class and corrects it on the blackboard, (2 minutes, 32 seconds), [lines 210-214].

2.3.2. The principal task

2.3.2.1. Production of statistical representations to order and obtain information

How does the teacher establish the task?

After introducing the issue of diet and using real data from the students, the teacher presents them with a handout of their snacks; one student points out that the snacks are disordered on the handout. The professor asks, "How can we order them?" [lines 40-42]. The children begin to share ideas and the professor registers on the blackboard the possible strategies that the students name. Then the teacher asks, "Why did I ask you all about snacks? What could we learn with the activity we will do today?" (The students predict *"Learn more about diseases", "Eat healthier"*). The professor affirms, "We'll see if today's activity can help you respond to that," and writes the challenge on the blackboard as a question, [lines 71-76] Then she reads the challenge to the students and re-reads it with different emphasis. "Each of your drawings is the snack that you like most [indicated by the child the previous day]. Can we order the data we have here to find out if we might have some disease? Is the activity very difficult?" ("*No!*" [in unison]) "You all are going to write the challenge in your notebooks and with the responses you all gave me [she points to the strategies written on the blackboard] you are going to try to order this [she shows the handout]," [lines 82-88].

The teacher proposes an open-ended problem, gives the students freedom in responding to it, implicitly suggests paths to follow, and also explicitly frames possible solutions based on the strategies proposed by the students. Through reiteration of the task, she forces the students to examine it. The challenge she proposes activates thoughts that are more complex than algorithmic thinking and demands cognitive effort and decisions in order to be able to respond. In this way, the teacher proposes a high cognitive demand task.

In the interview prior to the implementation of the lesson, Teacher C reflected: *"In the first lesson, our planning problems were evident. When we realized this* [the group about the failed lesson] *changed the planning so that the children felt closer to the problem and were motivated because they need to find something out; that is why we brought the context closer to the children, so that it would interest them, so that their own data would be important.*

(Interview with Teacher C) After implementing the lesson, she expressed: *"I think we improved the lesson. I am very surprised by the children's work; only one student said that we could order the data with a table, and another student named tables as a statistical concept. They can see a table, with divisions, that has columns, they understand it but they cannot verbalize it* [the table]." *(Interview with Teacher C.)*

How do students carry out the task?

As Stein and Smith (1998, 2000) point out, only when the teacher proposes high cognitive demands does the possibility exist that the students will mobilize high level cognitive demands.

One student drew a box and said, "*I'm going to make a box, and here I'm going to put what's healthy, and here what's not healthy.*" Another student drew five columns and said, "*I'm ordering... It's a box, and I'm going to put fruit, yogurt, milk, and so on...* [pointing to each column]". One student asks permission to cut the worksheet and put each food item in order [he indicates in his notebook that he will place them in columns]. In other notebooks two-column tables are observed with headings chosen and written by the children "*healthy food"* and *"unhealthy food"*, *"bad"* and *"good"*, *"healthy"* and *"bad"*, *"healthy food"* and *"junk food"*, *"unhealthy"* and *"healthy"*. One student draws a frame around the foods on the worksheet that she considers *"unhealthy"*; another girl counts the food items and crosses them out on the worksheet to not recount them. Without counting, other students draw a representative image in horizontal lists; others draw the image inside tables with headings and cells. Some students draw as many images as appear on the worksheet [graphical tables like pictograms appear]. One girl creates a two-column table with headings, in which she draws the image and beside it the corresponding count. A boy creates a pictogram without separating by category and without words. Only one girl creates a [frequency] table, using words and the respective count for each image. Other students make representations of nutritional pyramids with healthy and unhealthy food (the record of productions of the grade 3 students of Teacher C's lesson is found in Appendix V.6).

2.3.2.2. Justification of the representations

How does the teacher establish the task?

The teacher, after having given time for free individual work, asks, “Can anyone tell me how they thought to order the snacks [that were in the handout]?” One girl responds, “*I did it with numbers and I separated healthy food and unhealthy food. I counted them.”* Does anyone else want to share how they did it? “*I made a table and put what's unhealthy and what’s healthy.”*

The teacher summarizes, “The majority ordered what was in the handout in healthy and unhealthy [...] you all separated your snacks because they are yours. What did you put there? How did it turn out in your notebook?” The teacher invites three students, two girls and one boy, and asks them to take their notebooks and write on the blackboard what they have done in their notebooks. "You all did well ... how you thought to order this disordered information, but I want you to see what your classmates did, because what is here...” Some students try to complete the idea "*is the same or is different..."* [lines 196-209]. Do you see any differences between these strategies and what you did? Which do you think is the best option for responding to our challenge [the teacher points to the three strategies the students wrote on the blackboard]?

The teacher establishes a high level cognitive demand by listening to the students’ strategies and summarizing. In inviting three students with different strategies to the blackboard, the teacher allows the most able students to provide model solutions. The teacher repeatedly pressures students to give justifications, explanations, and meanings through questions, comments, and feedback.

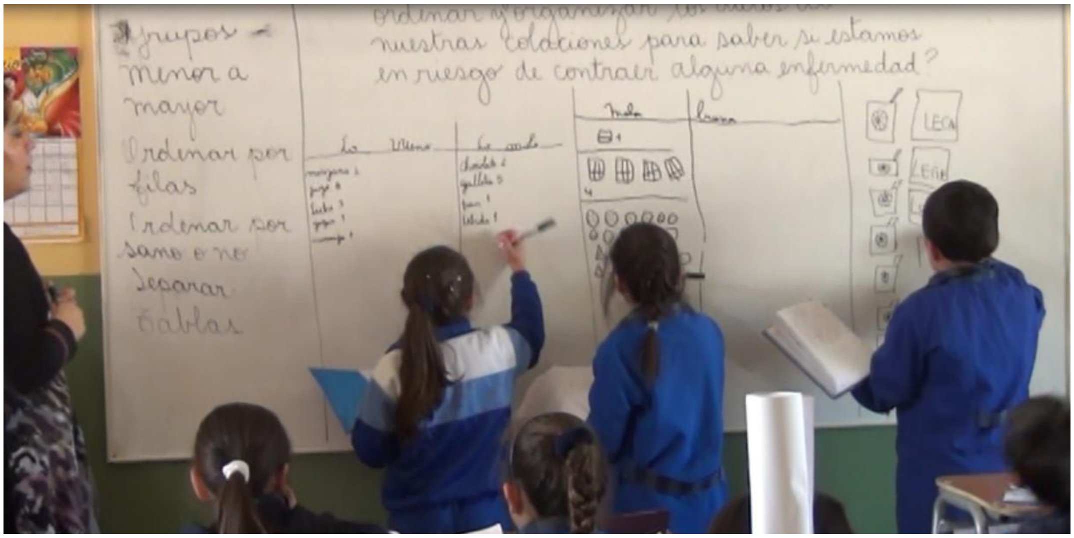


Figure 59 Representations produced by three students, from left to right: table with data and counting; table with icons and counting; and pictogram.

How do students carry out the task?

The students name what they observe in the representations on the blackboard, *"here we see order", "tables", "drawings, numbers, sums..."* [lines 190-194], and the majority of the students choose as the best option to respond to the stated challenge: the [frequency] table with words and counts of the elements.

In the following, some evidence is shown of the student productions and the characteristics of tabular representation are summarized:

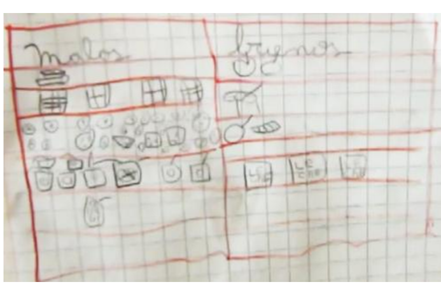


Figure 60 Table for a variable and its categories, with data represented by icons without counting.

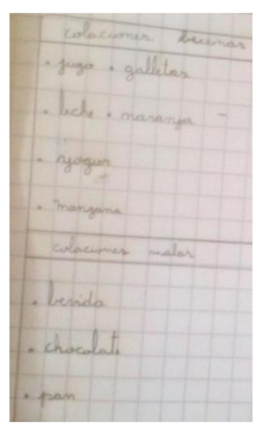


Figure 61 Table for a variable and its categories, with data represented by text without counting.

Figure 62 . Graphic table for a variable and its categories, with data represented by icons without counting.

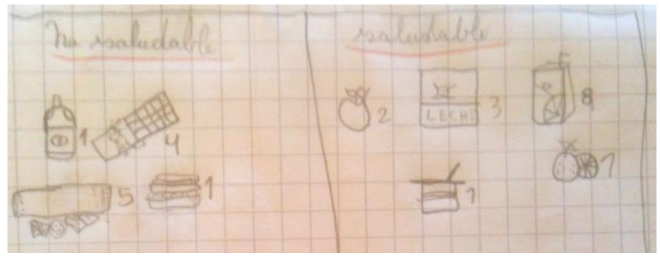


Figure 63 Table for a variable and its categories, with data represented by icons with counting.

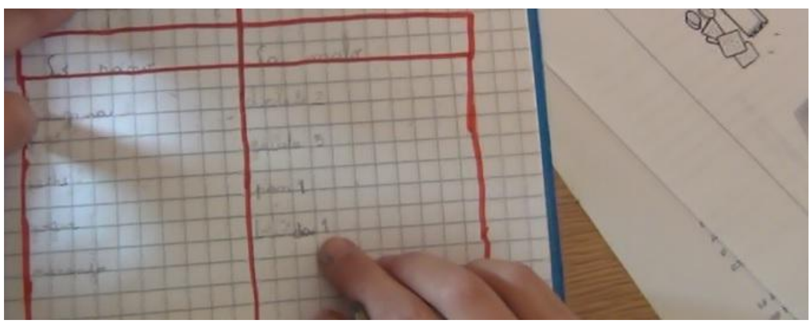


Figure 64 Table for a variable and its categories, with data represented by text with counting.

Table 15 shows a segment of the lesson that represents the general behavior of the teacher in the lesson implemented and the factors associated with the lesson's cognitive demands in its second implementation. In the lesson implemented, it was frequently observed that the teacher maintained cognitive demands; in the chosen segment, there are twelve items maintaining cognitive demands versus two decreasing. Appendix V.7 shows a comparison chart of the first and second snack lesson according to the level of cognitive demands in the entire lesson.

Table 15
Characterization of a representative segment of the second lesson with respect to factors associated with maintaining (M) or decreasing (D) high level cognitive demands.

	Factors in the second lesson
M	The teacher gives adequate time for exploration
M	The teacher provides structure is for student thinking and reasoning.
M	The teacher proposes the task based on students' prior work
M	The teacher listens to the students' strategies and summarizes
M	The teacher allows students to observe the different strategies of their classmates.
D	It is the teacher who talks about the students' strategies and summarizes them
M	The teacher looks after student reasoning
M	The teacher and the most capable students provide model solutions
M	The teacher makes connections to the students' prior knowledge
M	Teacher makes frequent conceptual connections.
M	The teacher pressures the students by responding to questions with more questions
D	The teacher shifts the emphasis from meaning, concepts, or understanding to the correctness or completeness of the answer

As mentioned, the coding of the factors was validated through an external analysis by two more judges, who also coded the lesson as having more maintenance of cognitive demands, which indicates the constant behavior of cognitive demand that Teacher C maintained.

Table 16 shows the evaluations of Teacher C's lesson (Appendix V.8 includes the transcriptions and details of the evaluations).

Table 16
Frequency of appearance of cognitive demands in the complete lesson, according to the author, based on consensus among two researchers and an external judge.

Maintenance of demand	Researcher1	Researcher 1 and 2	Judge 1
M1	10	6	25
M2	0	0	0
M3	10	7	4
M4	22	22	5
M5	4	1	5
M6	9	6	1
M7	0	0	0
total	55	42	40

Decreasing demands	Researcher 1	Researcher 1 and 2	Judge 1
D1	1	1	0
D2	8	6	2
D3	0	0	0
D4	0	0	0
D5	0	0	0
D6	0	0	0
total	9	7	2

The types of cognitive demands and their maintenance or decrease observed in the teacher in the second implementation of the lesson are diverse. Of the seven factors associated with maintaining cognitive demands, five were observed. Although in this study we are not interested in specifying the type of demand maintenance or the type of decrease of the demand, two types of demand maintenance were not observed, neither by the expert researchers, nor by the external judges: demand M2 (give students the means to monitor their own progress) and the demand M7 (give students sufficient time for exploration, neither too much nor too little), this absence of the factors of maintaining cognitive demands type 2 and 7 independently coded by four observers is an indicator that the category coding system works.

3. Discussion and Conclusions of Study 2

To understand how the teacher maintains the level of cognitive demand, it is necessary to understand what changes and how the lesson improves.

What changed?

Based on the discussion of the teachers who make up the lesson study group that made the “snacks” lesson plan, it is possible to confirm that in the first implementation of the lesson, getting the students to work on the problem was not achieved: there was a constant decrease in cognitive demands by the teacher who implemented the lesson, the students did not understand the problem, the teacher maintained a dominant role that was a detriment to student participation, and the snacks material used was too much (in number of worksheets) and was only presented on the blackboard.

In the second lesson -implemented by Teacher C-, the students understood the problem and the task; additionally, each student was given a worksheet from which the snack data could be obtained.

How did the lesson improve?

Learning from others and from oneself

As stated, once a week, during four weeks, five teachers from the statistics lesson study group met for two hours to plan and implement the snacks lesson. These sessions allowed them to interact in the planning of a lesson on statistical data analysis. In all the sessions, the teachers improved and discussed the lesson plan. Based on the lesson study group’s discussion, we became aware that the teachers, of their own accord, created on line meetings, not face-to-face, that allowed them to discuss and adjust the lesson they were planning together and that some of them were implementing.

Teacher C was able to observe *in situ* the first implementation of the lesson, the students' difficulties, and the execution of the first version of the lesson plan created together with the other teachers; this contributed to developing improvements to the lesson, especially specifying the central question and designing new material: the individual handout for the students.

Building a deeper understanding of statistics in the lesson study group

The researchers acted as monitors, intervening in the lesson study group to center the work on the statistical objects in play, and, in doing so: develop understanding of the statistical objects and implement a SRLE learning environment; increase the teachers' ability to explore and learn using data; develop statistical argumentation around a central question; and anticipate and understand the students' reasoning.

> "*I think that* [of the five teachers in the lesson study group], *I never felt that the opinions were destructive or meant to criticize in a negative way the execution of the planning that was created together." (Interview with Teacher C.)*

The gaps in the teachers' understanding of statistical concepts, the students' possible errors, and the anticipated responses that they would give to student questions to help them connect and interrelate concepts are part of a teaching focus that the teachers are not accustomed to.

> *"I think that we lacked* [in the planning] *better preparation for the children's questions and being able to specify the statistical nature of what was learned." (Interview with Teacher C.)*

The discussion within the lesson study group

The questions or doubts stated by the teachers in the group allowed them to debate, specify, and understand the difference between data and information; the differential characteristics between a list and a table; how to intervene in their students' productions to help them to "see and question" a representation created with some errors; the relevance of using real data of interest to the students; the importance of centering the development and the productions on a central question that requires data and motivates planning, analysis, and conclusions that respond to the central question of investigative work in statistics lesson.

The researchers applied pressure to promote questioning the teachers' statements about the implementation of the lesson with respect to the lesson plan actually implemented. The discussion of the teachers in the lesson study group about the first implementation of the lesson and the re-reading of the lesson plan allowed the teachers to confirm that the lesson had not worked, that the planning had not been successful. In the beginning, the teacher who implemented the first lesson and Teacher C thought that the lesson had worked, because the teacher presented the problem and the lesson goal, the students worked in their notebooks, and the teacher had finished and "closed the lesson". The confirmation of the failure of the first lesson came out of the discussion of the lesson study group, when the teachers checked the implementation against a reading of the lesson plan regarding the column titled "evaluating the progress of the lesson". This reading made it clear that the planning had failed: the children were not motivated by the presentation of the problem, the material created by the teachers was difficult to manipulate (because the worksheets were hung on the blackboard and copying from far away was difficult for the children; and there were many worksheets, about 50), issues that the teachers considered key to the success of the lesson. In evaluating conscientiously the lesson as it was actually implemented, the remade the lesson plan (see Appendix V.3), principally regarding the motivating introductory theme, the data delivered -in format and quantity-, the central question and the exposition of the statistical representations by the students; all of this resulted in maintaining high cognitive demands.

Understanding the different strategies deployed by the students, that is, the teacher becoming aware of students' reasoning, is a fundamental component of the SRLE model for teachers (Pfannkuch & Ben-Zvi, 2011), and this is revealed in the care the teacher made clear in maintaining cognitive demands. Another question of interest, for the learning of the teachers in the group as well as of the students for whom the lesson will be implemented, was the emergence of the frequency table as a cultural and meaningful construction. This table emerged in the various representations created by the 38 students to order the data about their snacks, among which we observed: simple lists with and without counting; simple tables with icons instead of words and repetition of icons as an expression of counting; tables with icons for categories and counting; tables with categories in words and counting; graphic tables, that is, tables with icons repeated in columns; among other things. All of the representations were useful to a greater or lesser degree for completing the task of classifying and representing the data; the interesting part of the proposal that this study describes is that in an environment designed to promote reasoning, the SRLE model, all of the strategies are respected, as they respond to the problem presented, and the frequency table is chosen as the best by the students themselves, considered by the children as the tabular representation that provides the best understanding among all those presented [lines 196-209].

196. Teacher C: I ask the little ones, of the three that are here, which do you think is the best option for responding to our challenge...
197. Students: number 1, number 1, number 1, number 3...
198. Teacher C: this does not mean that what the rest of you did is wrong... we are seeing which of these really helps us to respond to what I asked you... kids, then... Benjamin... which of these three do you think is the one that is most useful to us...
199. Students: number 1! [the students choose the frequency table with the snacks written and with counting]

Theory of teaching & learning in action

This study allowed us to approach the question of how a primary school teacher organizes a data analysis lesson, using the implementation of learner-centered teaching focus that is socio-constructivist in nature, the SRLE model -Statistical Reasoning Learning Environment- that we confirmed favors the development of deep and significant understanding of statistics and promotes high cognitive so that students can reason statistically "doing statistics". The lesson implemented, expressed in the lesson plan created, was principally founded in data analysis activities that, through the teacher’s management, gave students opportunities to: explore, observe, notice relations, think and reflect, share and debate with their classmates regarding different representations of data -tables and graphs- that they produced, providing structure and managing the discussion and statistical argumentation to justify and communicate results, as Ben-Zvi and Sharett-Amir (2005) and Estrella and Olfos (2012) have demonstrated at the school level.

At the beginning of the study we also asked ourselves about the characteristics of teaching activities whose goal is an understanding of data analysis. In this regard, the lesson was characterized by creating a learning environment in the mathematics classroom, achieving that students reflect on what they are doing, facilitating communication of the ideas produced in the classroom, and, through discussion and argumentation, being able to evaluate level of understanding of the data analysis concepts the students achieved.

> *"I was very surprised because the children that volunteered to explain their strategies are not the most outstanding mathematics students. The work in the* [teachers] *group was very enriching. The lesson we implemented was useful to me to see how students learn. I must begin to work more openly with the children, not limiting them by giving them everything ready-made. Our children are very capable." (Extract from the interview with Teacher C).*

The lesson study group, as an instance of professional development, constituted for the teacher a new way of seeing and acting in the classroom. It allowed her to make her understanding of statistical objects and their teaching. Also, as she recognized, she was surprised to realize that her role could be different than what it usually is and in doing so obtain greater benefits for the students. The interaction with her peers and the researches participating gave her new perspectives and an audience different than her students, an audience that can refute her, agree with her, and even collaborate in creating proposals for activities and tasks for the lesson.

STUDY 3: WORKINGS OF A TAXONOMY OF TABLE COMPREHENSION

1. INTRODUCTION

Faced with educational system's current challenge of statistical literacy, an particularly the necessity of having a teaching sequence for dealing with tables in the school curriculum, the present study focuses on characterizing the processes of understanding and in classifying the processes of reading tables. The questions that guide this study are: What are the cognitive demands that tasks with tables present? Is there a gradation of understanding these tasks? To respond to these questions, a proposal of levels of understanding tables is developed and its working is studied.

2. DEVELOPMENT OF A TABLE TAXONOMY AND ANALYSIS ACCORDING TO THE DELPHI METHOD

2.1. Introduction

This study deals with the questions proposed by Friel, Curcio, and Bright (2001) with regards to the necessity of paying more attention to tables as a representational tool and a data organization tool in the progression of the curriculum's development. To do so, a taxonomy of table understanding that captures, on one hand, the processes of understanding according to the purpose of the tasks, and, on the other hand, the processes of reading tables' structural components, was developed and its workings were verified.

The goals of the study are (1) develop a taxonomy of understanding tables according to the table structure and the role of the subject according to the purpose of the associate task, and (2) study the degree of consensus of the taxonomy with the Delphi method to verify its functioning.

2.2. Methodology

Responding to the study goals, the methodology considers two phases, as shown in the methodological design in Figure 66.

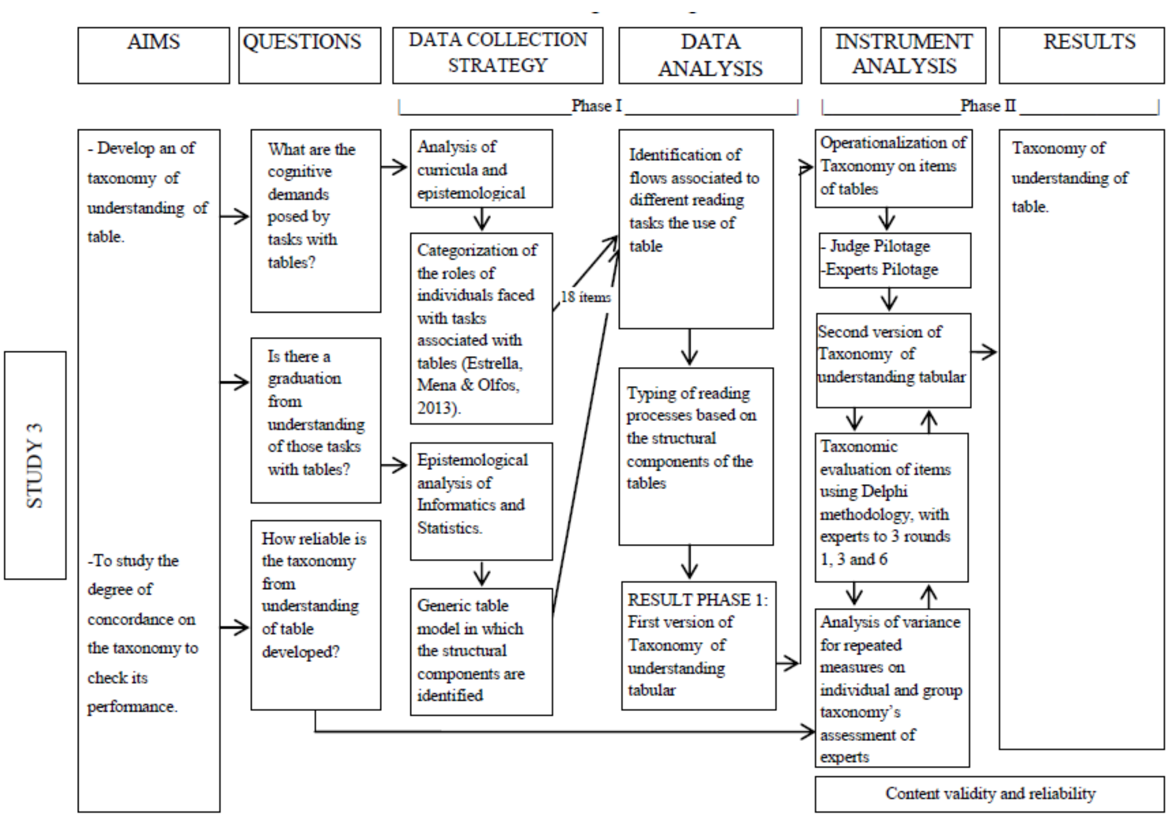


Figure 65. Methodological design of Study 3.

2.2.1. Phase I

The implementation of Phase I uses a selection of 18 items from international tests on tables, whose complexity is generally empirically in virtue of the performance of student populations. The battery of 18 items was taken from the grade 4 TIMSS (2003, 2007, and 2011), and from the Oklahoma Department of Education (ODE) primary school mathematics test (2008).

Instruments

The analysis of the items utilized the categorization of subjects' roles when faced with tasks connected to tables (Estrella, Mena-Lorca, & Olfos, 2014) and the generic table model, in which its structural components are identified (Estrella & Mena-Lorca, 2012).

Procedures

Phase one begins with the identification of the reading flows associated with different tasks using tables. Next, the reading processes are classified as a function of the table's structural components. From there, a table taxonomy is established, articulating the reading processes mentioned with the cognitive levels associated with subjects' roles according to the purpose of the task. Finally, examples are presented that guide the use of the taxonomy in classifying tasks.

2.2.2. Phase II

Phase II refers to a validation process in which the functioning of the taxonomy of table understanding is studied based on an analysis of its agreement with the Delphi method (Landeta, 1999).

Subjects

Thirty three students of the didactics in statistics course, students in the seventh semester of the pedagogy in basic education degree with concentration in mathematics, participated in the study. These students had knowledge of statistical representations and the official data and probabilities curriculum (MINEDUC, 2012)

Instruments

Taxonomy of table understanding with four levels, developed in Phase I, and a battery of 8 items from the TIMSS and ODE evaluations, taken from those used in Phase I of the study.

Procedures

First, a researcher, external to the study, contributed to improving the wording of the taxonomy's descriptors. Next, a pilot study was carried out with working teachers (n=18), in which the taxonomy was applied to the battery of 8 items. Finally, the 33 students from the didactics of statistics course categorized each of the 8 items according to the taxonomy and justified their choices, first individually, then in groups of three, and finally in groups of 6. The functioning of the taxonomy is studied with an F test as indicator of agreement, as established by the Delphi method.

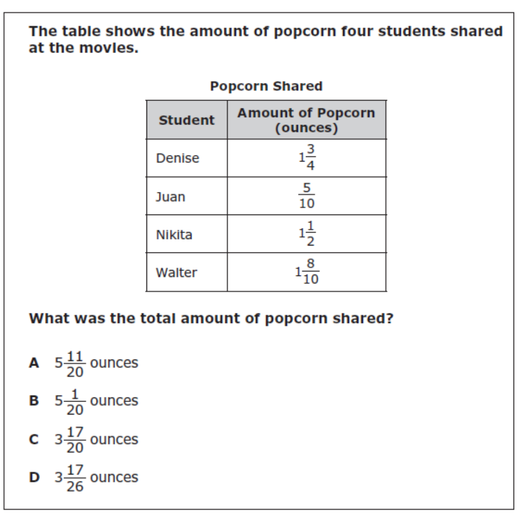
2.3. Results

2.3.1. Results for Phase I

Phase one begins with the identification of the reading flows associated with different tasks using tables. For each of the 18 items, a reading flow was obtained associated with each item's task, which revealed the importance of the table structure and the action of the subject's role (or, equivalently, the purpose of the task). A symbol system was developed for the representation of the flows, following the ideas of Janicki (2001) addressed in Estrella, Mena-Lorca, & Olfos (2013).

For items whose answer is found directly in the body of data or the headings, an empty rectangle was used (□); for items whose answer involves completing the table or using the body of data, a rectangle with a circle in its interior was used (◙); and for items whose response creates a representation outside of the table or is the result of more complex operations, a rectangle with a triangle in its interior was used (⧍). In cases in which reading was not necessary, a dotted rectangle was used; a normal rectangle for simple reading. (See Appendix V.9)

The four following examples show types of reading flow associated with certain items:
Example 1: Flow for a fractions table item. (ODE, 2008)

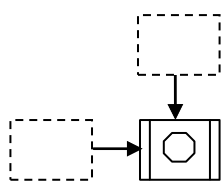

The table shows the amount of popcorn four students shared at the movies.

Popcorn Shared

Student	Amount of Popcorn (ounces)
Denise	$1\frac{3}{4}$
Juan	$\frac{5}{10}$
Nikita	$1\frac{1}{2}$
Walter	$1\frac{8}{10}$

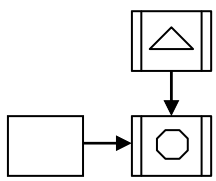
What was the total amount of popcorn shared?

A $5\frac{11}{20}$ ounces
B $5\frac{1}{20}$ ounces
C $3\frac{17}{20}$ ounces
D $3\frac{17}{26}$ ounces



The task considers the operation of addition, as it explicitly requires finding the total quantity. This task does not require any attention to the table's organization, that is, to the headings and cells -in particular- of the body of data (this is emphasize by the dotted rectangles in the reading flow diagram that signal the headings). As such, this is the most elemental reading process in the context of a simple task using a descriptive table. This is the level at which the task practically does not require a table and the user adds all the values contained in the body of data.

Example 2: Flow for an item using a table of hats (TIMSS, 2007)



There were 5 children at the park. Some were wearing hats and some were not.

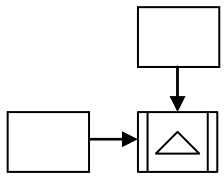
Girls	Boys
Maria was wearing a hat	Peter was wearing a hat
Megan was not wearing a hat	Chan was not wearing a hat
Mandy was not wearing a hat	

Complete the table to show the number of boys and girls that were wearing hats and were not wearing hats.

	Hat	No hat
Boys		
Girls		

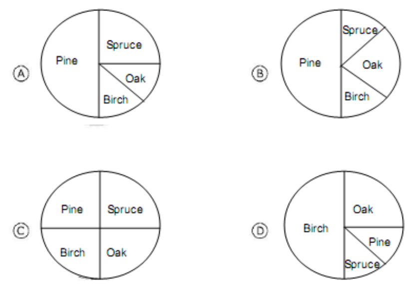
The task demanded requires using two data lists to complete a two-by-two table with given headings. The reading flow for the table includes the total reading of the lists and a count of the categories for a variable separated from the count of the categories of another variable, and then completing the body of data of the two-by-two table.

Example 3: Flow for an item using a table of trees (TIMSS, 2007)



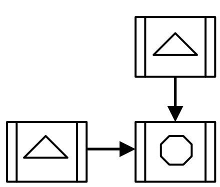
Type of Tree	Number of Trees
Pine	200
Spruce	100
Oak	50
Birch	50

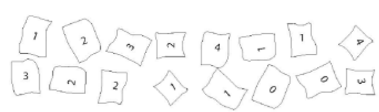
The table above shows the numbers of four types of trees growing in a park. Which of the following pie charts correctly displays the information shown in the table?



The task solicited requires converting from a table to another representation (pie graph). The reading flow for the table enters by way of both headings to reach the value contained in the cell in the data body. It is the numerical data which principally provides the area of the circular sector of the external graph being created (mentally or written).

Example 4: Flow for an item using a table of ballots (TIMSS, 2007)





Jasmin asked her classmates to write down how many brothers and sisters they had. She collected their answers and started to make a tally chart. She put in the two marks for the zeroes.

Complete Jasmin's tally chart.

Number of brothers and sisters	Tally
0	//
1	
2	
3	
4	

The task involves organizing external numerical data to carry out a count and then complete the frequency with counting marks, associating piece of external data with the category of the numerical variable. The reading flow includes identifying the heading and completing the body of data using an external calculation.

The second result of Phase I is the classification of the reading processes as a function of the table's structural components.

Based on the flows identified for the 18 items studied, four types of reading emerged. It was found that, in reading the cells, a *spot reading* was activated that pays attention to the cell's data. In reading a list, *sequential reading* was generated, which associates data with the heading (aiming at the local level of the list). In reading a table, *intensive reading* was activated, which relates headings with the body of data (aiming at the global level of the table). We also considered that an *extensive reading* could be generated, involving a table, its evaluation, and/or reach.

These four types of reading, already associated with tables' structural components, are linked to the role that the subject assumes according to the purpose of the task faced, and in consequence, are linked to the cognitive levels associated with these roles. In Table 17 these links are identified.

Table 17

Links between types of reading and the purpose and cognitive level of tasks related to tables.

Structural component	**Type of reading**	**Purpose of the task**	**Role of the subject**	**Cognitive level**
cell	point	record search complete	recorder searcher completer	low
list	sequential			
table	intensive	interpret build evaluate	interpreter builder evaluator	high
	extensive			

The third result of Phase I corresponds to a taxonomy of table understanding. The preceding relations allow us to establish taxonomy of table understanding with experimental support in levels of performance of an international sample of grade 4 students on items with tables from the TIMSS (2003, 2007, 2011) and ODE (2008) tests.

In the present taxonomy, in level 1 the subject reads cells without interpretation, focused on spot reading. In level 2, the subject reads lists, works with them and compares them in a sequential manner. In level 3, the subject focuses on an intensive reading of the table, globally analyzes the headings and body of data, and is able to interpret or build part of the table or other representations. In level 4, the subject focuses on an extensive reading of the table, also paying attention to the headings and the body of data; at this level the reading seeks the table's scope, as it connects the justification of its use in resolving problems, or a critique of the quality of its data, or the integration of context information, with the emission of judgments on the content and design of the table. Table 18 presents this taxonomy.

Table 18
Taxonomy of table understanding

Level 1	Reading cells (spot reading) to associate the data with the upper and/or lateral heading. The student works without comparing or interpreting with other data.
Level 2	Reading lists (sequential reading) to relate them with the data headings. The student completes, compares, or works with lists. This is a reading that does not need cells with data external to the list.
Level 3	Reading tables (intensive reading) to analyze all the headings and the body of data. The student builds or interprets using the table. Building includes formulating or completing another representation: algebraic, graphical, or tabular. Interpretation includes making comparisons between calculations contained in the table or between calculations produced using the table, and drawing conclusions using the table.
Level 4	Reading through the table (extensive reading) involves the table with its body of data and headings and includes justifying the solution to a problem base on the data in one or more tables. This also implies evaluating the table's level of communicability, or criticizing the quality of the data contained in the table, or drawing conclusions by integrating contextual information.

Phase I ends with the presentation of four cases that guide the use of the taxonomy in the context of classifying tasks.

Case A: Classifying an item in Level 1

The item "Fractions Table" in Example 1 asks the question, "What is the total quantity of goats?" whose answer does not demand an analysis of the headings or the meaning of the table; it is only concerned with the cells' quantitative information. The reading is sequential, works with the cells' content, and does not demand comparing, interpreting, or reading the headings. This item is classified in level 1.

Case B: Classifying an item in Level 2

The item "Table of hats" in Example 2 asks the student to complete the body of data in a table using two lists that are read sequentially and independently. The table provided does not require that the subject build the categories for the variables in play. This item is classified in level 2.

Case C: Classifying an item in Level 3

The item "Table of trees" in Example 3 involves an intensive reading of the table, including the body of data and its headings. It asks the student to use the table to build a graphical representation, that is, it includes transforming the table to another representation. This item is classified in level 3.

Case D: Classifying an item in Level 4

None of the 17 items studied is level 4. The item "Table of ballots" from Example 4 (which corresponds to the advanced performance level of the TIMSS) asks the student to read the table in an intensive reading, involves the entire table, and implies obtaining data from outside the table in order to organize data, make a count, and complete a table with marks equivalent to these calculations. We classify it as level 3. To classify it as level 4, we have proposed that the item could include a question like, "Sara found a ballot with the number 7. How would you reformulate the table to include this new data?" The solution to this new question leads the student to change the table's design according to the data, completing it with a new class "more than 5" or "other number" in the lateral heading and making a mark in the frequency list.

2.3.2. Results for Phase II

First, a researcher, external to the study, contributed to improving the wording of the taxonomy's descriptors. He analyzed the wording of each item and compared it to the description of the associated taxonomic level, contributing to the wording, making terms more precise, and making the corresponding rubric clearer.

Next, following the suggestions of Landeta (2006), a pilot study was carried out with working teachers (*n*=18), which contributed to creating the answer format in the questionnaire of 8 items and creating a protocol for administering the Delphi method.

Next, the Delphi method was implemented with 33 subjects, students in the seventh semester of the pedagogy in basic education with concentration in mathematics degree. They were given the category of experts, as they are advanced students with a relevant concentration who were taking the didactics of statistics course, and they were asked to judge times from a statistical table for grade four students. During periods of approximately 75 minutes, the experts, in person, classified, in writing, each of the 8 items on tables according to the taxonomy's levels, providing justifications for their choices. First they classified them individually, then in groups of three, and finally in groups of six. The three stages of consulting on the same taxonomic task favored the experts' responses focusing on the information that came up in the group discussion. The interaction of the experts led to a unification of the arguments about the choice of taxonomic levels, allowing them to reconsider or maintain their criteria. The group of experts was stable: of an initial 36, 33 (92%) participated in the entire process. The time between rounds was 15 days, and the duration of the entire process was less than two months. The coordinator collected the experts' commentaries, which contributed to making the wording of the taxonomic descriptions more precise.

The coordinator is the academic responsible for the didactics of statistics course that the experts were taking. He has a thorough command of the taxonomic concepts and knows the experts and there motivations and was the person who maintained the study's continuity.

Finally, statistical results were obtained for the application of the Delphi method. The questionnaire of 8 items on tables that was given to the experts to evaluate allowed each individual -and later each group- to rate each of the items with a score of 1 to 4 based on the rubric of the table taxonomy. A repeated measures variance analysis was implemented for the experts' individual and group taxonomic evaluations.

Table 19
Repeated measure variance analysis

Efectos	Gl Num	Gl Den	SC Num	SC den	Razón F	Valor-p (p<0.05)	Eta^2_G
Intercepto	1	7	143.6	9.340	107.606	0.0000167*	0.926
Evaluación	2	14	0.1	2.004	0.265	0.7710695	0.000

A low F-value and a p-value greater than 0.05 were obtained, which allows us to assume that there are no significant differences among the different evaluation judgments. That is, the judgments of the individual evaluators and the Delphi groups (grouped as the evaluation variable) do not differ from one another.

That is, in general the taxonomies are applied in a similar way among individuals as well as among consensus groups. In consequence, the taxonomy is sufficiently reliable to be applied by professionals in the area.

3. Discussion and Conclusions of Study 3

This study proposes: (1) to develop a taxonomy of understanding tables according to the table structure and the role of the subject according to the purpose of the associated task, and (2) to study the degree of consensus of the taxonomy with the Delphi method to verify its functioning.

The principal result obtained is the taxonomy of table understanding, which was created by identifying the reading flows associated with different tasks using tables and by classifying the reading processes as a function of tables' structural components.

The Delphi method and its statistical analysis provide content validity and reliability for the taxonomy to be utilized by experts in the area in the context of the chosen tasks.

To establish level of complexity in dealing with tables, the types of reading activated by tasks were looked at. At the same time, subjects' roles according to the purpose of the tasks the faced were considered.

The taxonomy of table understanding developed includes types of reading associated with a local area, a part of the table, or the entire table. Specifically, for level 4, extensive reading of the table involves a critical reading of the table and its repercussion; for level 3, intensive reading implies a global reading of the table; for level 2, sequential reading means reading a list of data that can do without some cells and/or headings; and for level 1, a spot reading of cells or lists that does without the headings entirely.

Kaput (1991) in his writing on notations and representations as mediators of constructive processes, points out that a factor related to the power of a notation is its degree of abstraction. We consider that in order to understand tables in depth, on must distinguish between their material existence as a record of semiotic representation and their existence as a cognitive and cultural object. Cognitively, tables are in a hierarchy superior to lists that establish spot, sequential, and reticulated binary relations, as tables allow for establishing multiple relations. As a concrete notation, tables have characteristics of internalization and externalization. This study allowed us to identify internal processes of searching, interpretation, and evaluation, and external processes of building, completing, and recording.

Limitations. One of the characteristics of this study is the use of the taxonomy on the same items that were used to create it. Some of the characteristics of each level of understanding of tables were reconciled with the performance levels on the TIMSS (2003, 2007, and 2011); that is, the reading flows and the roles of the subject faced with the task were based in 18 items from international tests. While on one hand, the use of the same battery of items gives it coherency, on the other hand, it restricts its range of application. As such, we will return to this point in Chapter VI.
This study is inspired by the work on taxonomy of graph understanding (Curcio 1987, 1989). In the teaching guidelines provided by Friel, Curcio, and Bright (2001), the authors maintain that, in exploring representations of data, tables should be considered the most conscious form for students, given that they are used as tools for organizing data as well as for displaying it visually.

Although this study has tried to differentiate itself from graph taxonomy, based on the premise that tables are not graphs and respond to different needs, in order to help mental association, the taxonomic levels identified for understanding tables are linked to the categories of Curcio (1989) and Aoyama's (2007) graph taxonomies, recognizing, of course, the theoretical differences on which these are built. In this respect, level 1 of our taxonomy corresponds to “reading in the data"[28], although without interpretation, and focusing on cells and activating spot or sequential reading. We assume that the cognitive demand of working with part of a table should be less than when working with the entire table. In our taxonomy, level two works with and interprets only lists from the table, and level 3 includes an analysis of the table. So, levels 1, 2, and 3 are associated with the levels proposed by Curcio (1987), and level 4 corresponds better to the critical and hypothetical levels of Aoyama's (2007) taxonomy, as there is an evaluation of the table and relevance is given to the situation in context.

Following the findings of Gabucio et al. (2010) the taxonomy developed here adapts itself to the physical structure of tables; our contribution has been to introduce the focus of localization that regulates the reading and the purpose of the associated task.

An investigation by Ben-Zvi (2004) carried out with grade 7 students with fluidity in representational aspects of EDA, shows that the students make use of tables and graphs when they do data analysis in order to find generalities (global vision) an outliers (local vision). This finding is considered in building the taxonomy of understanding tables with respect to readings, some of which address local aspects (reading cells and lists) and others of which include a global reading of the table.

At the same time, with respect to the purpose of the task, the study allowed us to seek out and specify more purposes than those that had been indicated initially. We established the action of reading tables as a crosscutting and basic action, identified a group of actions related to tables: record, search, and complete, and differentiated them from another group of actions: build, interpret, and evaluate.

In summary, the taxonomy of understanding tables integrates the physical components of tables (rows, columns, cells) with the reading of data -arranged in lists or cells in the body of data- and the variable categories -found in the margin-.

The purpose of this study is not to deliver a taxonomy of table understanding to be applied exactly for an item or situation of table learning, but rather to have an instrument that allows us to anticipate the level of difficulty of tasks using tables, and in doing so, provide a tool for

[28] Level 1 proposed by Curcio.

configuring a didactic sequence that considers, for example, the list as a basic constituent element to begin test items, text activities, teaching sequences, or curricular units, and to recognize that tables posses a physical structure of localization and semantic content, which allows us to work with them, for example, concatenate them. This knowledge about tables should favor tables being considered as a learning and teaching object in the first years of school, and students gradually taking possession of them as cognitive tools, and teachers and curriculum designers addressing their configuration, properties, and operations.

STUDY 4: ANALYSIS OF THE CONCORDANCE BETWEEN GRAPH TAXONOMY AND TABLE TAXONOMY.

1. INTRODUCTION

This study seeks to determine if the proposed taxonomy of table understanding from Study 3, above, is similar to an existing taxonomy about graph understanding. The question that guides this study is, "Are the levels of graph understanding the same as the levels of table understanding?"

To respond to this question, an analysis was carried out that seeks to establish the degree of concordance between the evaluations assigned using the taxonomy of graph understanding and the taxonomy of table understanding we created in Study 3. In the following, we try to establish the agreement that the judges indicate in assigning ratings to eight items about table representation.

2. METHODOLOGY

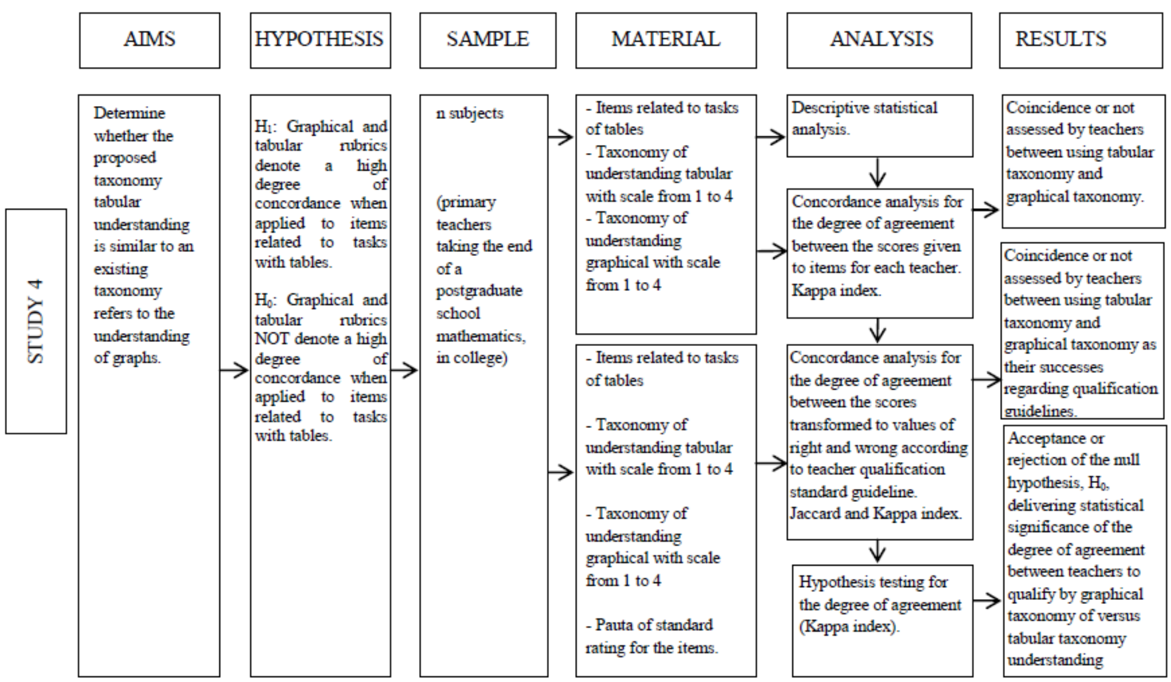


Figure 66. Methodological design of Study 4.

This study uses quantitative analysis that is described in detail in the methodological design, see Figure 67.

2.1. Subjects

The items were responded by a total of 18 working teachers; all of them belong to a primary school that was participating in a program of continued education of the Pontifical Catholic University of Valparaiso during the beginning of 2013.

2.2. Design

The design consists in a set of items taken from the TIMMS and the ODE (see Study 3), about tasks associated with tables. The technical aspiration of the study was to evaluate the qualities of the items and not the subjects' performance. To do so, the participating teachers applied a taxonomy that was originally aimed at evaluating graphs, under the supposition that its application was indiscriminate for graphs and for tables. Additionally, the same teachers evaluated the same items, but using a taxonomy created specifically to determine the attributes of tables. The scores assigned correspond to a 4-point scale in both cases.

As such, there are two evaluations using different taxonomies, but in hopes of obtaining the same results, given that it is assumed that graphs and tables share the same indicators of pertinence and adequacy.

2.3. Analysis

The analysis applied consisted in an initial description of the results, using descriptive statistics and comparing the scores obtained by the teachers in the evaluation using the graph taxonomy and the rubric for tables. Later, a concordance analysis was carried out in which the degree of agreement between the evaluations made by the teachers was established.

The statistical options for determining agreement are varied and depend on the intent of the study. In this case, in the search for concordance in the assignation of scores to the items using both taxonomies -graph and table-, we aim to establish that the taxonomies in one case or another are different and cannot be used indiscriminately to evaluate table items and graph items.

As such, the statistical hypotheses indicate:

H_1: Graph and table taxonomies show a high degree of concordance when they are applied to items about tasks with tables.

H_0: Graph and table taxonomies **do not** show a high degree of concordance when they are applied to items about tasks with tables.

The analysis used is supported by two different approaches. Given that the scores that could by assigned by the taxonomies are from one to four points in both the table and the graph models, and that also there is a standard criteria for the value that these tables should adopt for each item evaluated, we proceeded in two distinct manners.

First an ordinal variable concordance analysis was made, which involved maintaining the scores originally assigned by the teachers and establishing if there was concordance among the evaluations as obtained. This means that the teachers' scores were not evaluated as a function of the expected taxonomy, which would have lead to transforming the original scores as a function of their accuracy or error with respect to the correct or expected response (standard scores model), but instead with the original scores a matrix was created with ordinal values and a first estimation of concordance was obtained.

Next, a concordance analysis for dichotomous variables was made, which implied transforming the original scores to values of accuracy and error for each teacher. The accuracy and error of the teachers was determined by comparing the scores they assigned with the model of standard scores that had been previously created. In this way, a matrix of scores of zeros and ones was obtained, indicating when the teacher scored the same as the standard model and when no. Using this matrix, a second approximation of concordance was calculated.

In the first case, the analysis applied corresponded to the Kappa index for ordinal variables, a special case of the Kappa index for presence/absence (Fleiss, 2000; Landis & Koch, 1977). In this case the index is equal to the mean of the $n(n-1)/2$ Kappa indices obtained by each of the evaluators or judges. The los values are omitted. It is a weighted Kappa, whose weights (the square of the absolute values) are obtained from the variable weights (Conger, 1980).

In the second case the Jaccard index was applied as the principal index and Kappa for comparison; the reason for this is that, in the case of Jaccard, the calculation of agreement is made based on the present conditions, and absent conditions, or when the attribute is not present, are discarded. In the context of presence/absence tabulated with ones and zeros, the Jaccard index only considers the cases where there are ones, and all of the cases where only zeros appear are discarded. Put another way, it does not consider the concordance in which two judges evaluate an attribute as absent. In practice, this means calculating the degree of agreement as a function of the presence of an attribute detected by both judges or evaluators and the presence of an attribute detected by only one of the judges or evaluators, discarding the concordances in scoring the attribute as absent.

It should be kept in mind that both the Kappa index and the Jaccard index range from 0 (null agreement) to 1 (total agreement)

3. RESULTS

The results are shown according to how the variables were treated, first the descriptive and then the concordance treatments of the ordinal variables, later the dichotomized variables are presented in the same order.

3.1. Analysis based on ordinal scoring

As seen in Table 20, the scores given using the table taxonomy tend to be higher than the scores obtained by applying the graph taxonomy.

Table 20

Statistical summaries of the ordinal scores given to the items

	Taxonomía tabular valores ordinales					Taxonomía gráfica				
Ítems	**Media**	**DT**	**Mediana**	**Mínimo**	**Máximo**	**Media**	**DT**	**Mediana**	**Mínimo**	**Máximo**
Ítem 1	1.56	0.86	1	1	4	1.22	0.43	1	1	2
Ítem 2	2.72	0.89	2	2	4	2.06	0.64	2	1	3
Ítem 3	2.67	1.24	3	1	4	2.44	0.92	2	1	4
Ítem 4	2.44	0.86	2	1	4	2.33	0.84	2	1	4
Ítem 5	3.22	0.73	3	2	4	2.33	1.03	2	1	4
Ítem 6	1.44	0.51	1	1	2	1.39	0.61	1	1	3
Ítem 7	2.78	0.88	3	1	4	2.44	0.70	2	2	4
Ítem 8	2.33	1.08	2	1	4	2.06	0.87	2	1	4

Apparently, the application of taxonomies specific to graphs tends to underestimate the information presented in table form. The scores given using the table taxonomy offer somewhat higher scores when taxonomies specifically adapted to this type of item are used. In particular, these differences can be appreciated in items 2, 5, and 8.

Also, if the standard deviation (SD) is taken into account in comparing the taxonomies, we can also appreciate that the dispersion is less for the graph taxonomy, which indicates less flexibility and a more homogeneous evaluation by the teachers in scoring each item. An exception occurs with item 5.

Additional guidance is offered by the means of the items as a function of taxonomy. We observed that when the evaluation is made with the taxonomy for tables, the mean of various items is greater than a score of 2, while when graph taxonomies were used, the score for these items was less than 2 points. This confirms the results shown by the means and standard deviations.

With the ordinal data it is possible to calculate an adapted concordance index for this type of variable. As mentioned, the Kappa index for ordinal data is applied (Conger, 1980). The values given by the teachers for the items are not modified, nor are they corrected according to the prior criteria model. The aim is to determine whether applying these taxonomies a degree of agreement is obtained that allows us to state that the application of one or the other is indistinct. Table 21 shows the values obtained.

Table 21

Kappa agreement indices for ordinal scores

Índice kappa ordinal	
Ítems	**Valor**
Ítem 1	0.04
Ítem 2	0.20
Ítem 3	0.16
Ítem 4	0.44
Ítem 5	0.14
Ítem 6	0.45
Ítem 7	0.33
Ítem 8	0.29

Even when the original scores provided by the teachers have been respected, we can see that the Kappa values in general are low, never greater than 0.45. While there is no unique reference to interpret agreement values in social sciences, Landis and Koch (1977) indicate that agreement greater than 0.75 is excellent, from 0.4 to 0.75 is good, and less than 0.4 is poor. The average Kappa agreement was 0.26.

With these results it is possible to state that what the teachers evaluate utilizing the table taxonomy does not correspond to what they evaluate using the graph taxonomy.

Although there is some agreement in some items, it remains to be established what characteristics these items present that favor a greater agreement and what is happening with the items that do no achieve the same degree of concordance.

3.2. Analysis based on dichotomous scoring

The results obtained above are established as a function of the scores that the teachers actually assigned to the items. However, there is a prior criterion that indicates the score that should be assigned to each item. As such, the scores given by the teachers were compared with this criterion. When they coincided, it was assumed that the score given by the teacher was correct, and a one was assigned for each accurate score with respect to the criteria. When the scores did not coincide, a zero was assigned. In this manner, the scores were dichotomized. In this way, a second matrix of data of ones and zeros was obtained (agreement or disagreement of the teacher's evaluation with the pre-established scores) which was analyzed descriptively. The results are shown in Table 22.

Table 22
Descriptions of the dichotomous scores given to the items

	Taxonomía tabular valores dicotómicos					Taxonomía gráfica				
Ítems	**Media**	**DT**	**Mediana**	**Mínimo**	**Máximo**	**Media**	**DT**	**Mediana**	**Mínimo**	**Máximo**
Ítem 1	0.28	0.46	0	0	1	0.22	0.43	0	0	1
Ítem 2	0.56	0.51	1	0	1	0.61	0.50	1	0	1
Ítem 3	0.11	0.32	0	0	1	0.50	0.51	0.5	0	1
Ítem 4	0.17	0.38	0	0	1	0.11	0.32	0	0	1
Ítem 5	0.44	0.51	0	0	1	0.22	0.43	0	0	1
Ítem 6	0.00	0.00	0	0	0	0.06	0.24	0	0	1
Ítem 7	0.06	0.24	0	0	1	0.00	0.00	0	0	0
Ítem 8	0.44	0.51	0	0	1	0.44	0.51	0	0	1

In this case the results behave differently. Given that the table reflects agreement with the standard criteria, each item denotes a proportional mean, indicating the degree in which the teacher's evaluation coincides with the standard criteria. So, for example, in item 1, in the evaluation of the eight cases considered, the teacher responded in agreement with the criteria in a proportion of 0.28, or 28% of the time; in item 2, his evaluation agreed with the criteria in 0.56, or 56% of the time.

It can be seen that in using both the table taxonomy and the graph taxonomy, the proportion of agreement is modest, indicating in both cases that the teachers did not evaluate the items as expected.

Secondarily, the table of dichotomous data can also be analyzed as a function of the degree of agreement between the taxonomic evaluations carried out. As indicated, the concordance must be estimated using the dichotomous data that reflect accuracy, or correspondence of the teacher's scores with the prior criteria. Table 23 shows the corresponding values.

Table 23
Jaccard and Kappa agreement indices for dichotomous scores

Ítems	Índice de Jaccard Valor	Indice Kappa Valor
Ítem 1	0.13	-0.03
Ítem 2	0.62	0.43
Ítem 3	0.10	0.00
Ítem 4	0.25	0.30
Ítem 5	0.20	0.05
Ítem 6	0.00	0.00
Ítem 7	0.00	0.00
Ítem 8	0.60	0.55

As we said, the Jaccard index better reflects agreement with dichotomous data (accuracy/error, presence/absence, etc.) because it only takes into account the cells with values of accuracy, and as such provides a value for agreement regarding information really provided, ignoring the agreement that could be produced by information not provided by either evaluator or judge. That said, the Kappa indices obtained for the dichotomous data are also provided for comparison. With Jaccard, slightly higher indices are obtained, but in general they agree in the low level of agreement in scoring given by teachers using one or the other taxonomy.

As can be seen, the values of the Jaccard index, which also range from 0 to 1, as in the Kappa index, are not any more inspiring, despite the improvement that their calculation proposes. Only items 2 and 8 obtain a very good degree of agreement; the rest of the items reflect weak or even null agreement. The average agreement index was 0.24, similar to the average agreement in Kappa.

In the context of this work, it can be concluded that there would not be agreement between the evaluations made using the table taxonomy and the graph taxonomy. The scores given by the teachers differ and, consequently, the scoring taxonomies guide the scoring in different ways.

3.3. Hypothesis testing for the degree of agreement

The Jaccard index does not have an associated statistical test that allows for making a decision with respect to the significance of the degree of agreement achieved. The Kappa index, in contrast, allows for applying a statistical contrast that offers a Z score with an associated p-value. While it is true that the Kappa index offers lower indices, this is a limitation that makes the contrasts more conservative. However, we consider the differences between both indices to be rather negligible. Table 24 shows the corresponding results.

Table 24
Statistical testing for Kappa index

	Contraste del índice Kappa		
Ítems	Valor	Estadístico Z	Valor-p
Ítem 1	-0.03	-0.14	0.8882
Ítem 2	0.43	1.84	0.0661
Ítem 3	0.00	0.00	1.0000
Ítem 4	0.30	1.34	0.1797
Ítem 5	0.05	0.25	0.7998
Ítem 6	0.00	0.00	1.0000
Ítem 7	0.00	0.00	1.0000
Ítem 8	0.55	2.33	0.0196

The tests indicate that, with the exception of the evaluations given for item 8, the degree of agreement among teachers was not statistically significant, such that in seven of eight tests, it can be stated that there was no agreement or concordance in the application of the taxonomies. Consequently, it appears that both taxonomies perform differently in guiding the scoring that the teachers must realize.

4. Discussion and Conclusions of Study 4

Diverse authors have investigated regarding the abilities implicit in reading and understanding statistical tables and graphs, and they have defined various levels (Bertin & Barbut, 1967; Curcio 1987, 1989; Baillé & Vallérie, 1993; Friel, Curcio, & Bright, 2001; Aoyama, 2007; Shaughnessy, 1996, 2007). Generally, a taxonomy of understanding graphs is presented as if the levels could also be considered for tables (Arteaga et al., 2011).

In Friel, Curcio, and Bright’s (2001) research on understanding graphs and tables, the authors focus on tables linked to graphical representations, considering them only as an intermediate stage for the creation of graphs, a “transitional tool” (op. cit., p. 128). Given that the present study considers that tables allow for observing regularities and, as such, reflecting on data, Friel et al.'s assertion is taken into account and it is recognized that the taxonomy of understanding is only for graphs and should not be used indiscriminately for tables. The supposition admitted could be compared to Fried et al. (op. cit.), regarding recognizing a distinctive role for tables, they maintain that tables should be considered the most conscious form for students in the exploration of representations of data, given that beyond the visual display, the are useful as effective tools for organizing and representing data.

The study tries to respond to the dilemma, “Are the levels of graph understanding the same as the levels of table understanding?” In pursuing this, we sought to determine if the proposed taxonomy of table understanding behaves similar to the taxonomy for understanding graphs, associated with the authors Friel, Curcio, and Bright.

Statistical analyses were carried out according to ordinal scores and according to transformed scores, and a hypothesis test was done on them. The analysis based on ordinal scores allows us to state that what is evaluated by the teachers using the table taxonomy does not coincide with what is evaluated using the graph taxonomy. Based on the analysis of dichotomous scores, it is possible to state that the scores given by the teachers vary, and, as such, both taxonomies display different orientations. Finally, and based on the hypothesis test for the degree of concordance among teachers, this was not statistically significant. As such, the taxonomy of graph understanding guides scoring differently than the taxonomy of table understanding.

With the data gathered and the analyses made, it is possible to determine that the taxonomy of graph understanding is not similar to the proposed taxonomy of table understanding.

The relevance of this finding is the value that the taxonomic proposal of table understanding developed in Study 3 takes on, and that it empowers and encourages future studies and applications.

CHAPTER VI

Conclusions and Prospects

Chapter Summary

1. A BRIEF SUMMARY

Our work seeks answers to the basic question, "How do children understand tables?" To this end, we first realize a literature review about tables from their beginnings. We broaden and integrate knowledge about tables, carrying out a historical epistemological study and investigating some areas of application such as informatics, statistics, and its place in the curriculum.

The polysemy and complexity of statistical and mathematical meanings of the concept table motivated us to research the link between, on one hand, the structure of knowledge about tables created by the discipline and, on the other hand, the conceptual structure of tables created by students, especially the frequency table at the school level. In order to try to describe the initial level of conceptualization of tables that the students had, we adopt the Theory of Conceptual Fields (TCF). We adopt the TCF, particularly because Vergnaud pays attention to the progressive meaning of concepts that the subject forms through problematic situations, together with language and symbols, and because his theory values the implicit knowledge of students faced with a situation and focuses on reconstructing this knowledge to make it explicit.

Also, from a didactic perspective, this model gives the teacher the role of mediator, responsible for creating and designing the task adequate for activating semiotic schemes and expressions of a certain conceptualization; and it is the teacher who must present the students with the activation of schemes and help them to make the concepts and properties clear in the right moment, taking into account and using in his or her teaching the concept’s continuities and ruptures.

To design the table learning situation, a model of statistical education that promotes statistical reasoning (Garfield & Ben-Zvi, 2008) was adopted, as well as a perspective of the processes activate when representations are changed (Wild & Pfannkuch, 1999). As a didactic system, we have also looked at the teacher’s mediating acts through observation of the cognitive demands that they promote (Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2000).

With the support of these models, our investigation is a first exploration of primary school students' progressive mastery of the conceptualization of tables.

We carried out four studies to address our research questions.

Study 1 sought to respond to the questions: (1) How does the notion of tables emerge in students in the first years of school?; (2) How do students create meaning from data?; (3) What representations do students create when faced with a data analysis task?; (4) What is the thinking behind the representations that the students produce?; and (5) What levels of conceptualization to these representations reflect?

Study 2 tried to respond to the questions: (6) What are the characteristics of a teaching task aimed at data analysis?; (7) How does the teacher manage a primary school data analysis lesson?; and (8) How does the teacher maintain the task's level of cognitive demand?

Study 3 sought to respond to the questions: (9) What are the cognitive demands posed by tasks associated with tables?; and (10) What are the components of a hierarchy of table understanding?;

Study 4 asks (11) Are the levels of graph understanding the same as the levels of table understanding?

In the following we briefly describe the epistemological, cognitive, and didactic studies of tables, and the findings and conclusions of each one.

2. FINDINGS AND CONCLUSIONS FROM THE EPISTEMOLOGICAL STUDY

The study of the process of historical evolution of ideas about tables and their connotation as a tool that accompanies the development of human thought enriched our knowledge about tables and their didactic reach. Specifically, it delivered knowledge about the development of tables and their presence in different cultures as a tool for storage, calculation, and analysis in administrative, economic, scientific, and/or mathematical spheres. Tables constitute a useful tool for recording empirical data, ordering it, and creating information based on it. Tables have promoted knowledge creation (e.g. numbers and functions) and forged tools for formulating, transmitting, and utilizing knowledge expeditiously.

A review of prior research in the last ten years allowed us to reveal tables as a mathematical object with independent development that impact other areas of the discipline; their present rapid development as a mathematical object manifests the importance tables have for mathematics.

Tables' epistemological development allows us to observe them first as a *proto-mathematical object*, later as a tool for studying other mathematical objects, therefore a *para-mathematical object*, and only recently have they taken on status as a *mathematical object* to be studied in its own right.

The study allowed us to discern the epistemological evolution of tables in their double role of tool and object. A change in the use of tables and in the development of statistical thinking was provided by Graunt in the seventeenth century, when he used tables to analyze and classify data and to create new tables. These tables emerge from a model based on previously tabulated data in order to make predictions and scientifically document government decision making.

We agree with Pecharroman (2013) in that the definition of mathematical objects emerges from the expression of their functionality, the properties that allow their differentiation from other objects, and the relations that situate them in existing knowledge. The expression of these aspects creates a mathematical object and permits its definition.

Although we find table algebra dealt with as a mathematical object only in the last decade, tables as a mathematical object begin to take shape as such several centuries ago. Pecharroman (2013:130) maintains that mathematical knowledge is developed through new uses given to objects based on the functionality represented when "their meaning is broadened". In this regard, we can consider scribe schools, where tables were used as an extension of individual human memory, and in the same culture and geographical area, tables became a repository for the circulation of knowledge among communities of scribes. Similarly, we can consider the use of tables with differentiated roles for working with astronomical data, as a data repository, a means of calculating, or an analysis tool, table functionalities that possibly allowed Ptolemy to use tables as the quantitative representation of his model and at the same time as a tool for evaluating specific values in the model.

We also agree with Pecharroman (op. cit.) in that mathematical knowledge is developed through reinterpretation of an object (or creation of a new object) when a functionality that it represents in other contexts is perceived. In this regard, we can consider the tables of anthropometric measurements created by Quetelet in 1833, which allowed him to infer an index that associated the variables weight and height in a simple manner; this index would later be divulged in a scientific article in 1984 and widely cited[29], which made public the good behavior of the Quetelet index (or body mass index, BMI), and from which anthropometric tables are created today.

Additionally, we also agree with Pecharroman (op. cit.) in that mathematical knowledge is developed through modifying objects (or creating others) due to discovering errors. In this regard, we can consider the moment when Babbage, 1829, communicates some common errors in many logarithmic tables and begins to promote the potential of calculating machines, which leads to the creation of automated tables in 1849, tables which are obtained more accurately and more quickly.

Today, tables occupy simultaneously or partially all of the roles that we have reviewed, such as a storing data, facilitating calculations, and/or analyzing data. Tables, although generally used only as auxiliary representations that help to use other records, posses their own rules of use and, as we have described, allow for organizing information and producing new knowledge. In particular, statistical tables have promoted the use and development of abductive reasoning in the Peircean sense. New technologies allow us to produce tables and work within one table or among multiple tables; now with a defined structure and properties, tables take their place as a mathematical object.

[29] Garrow and Webster, 1984.

3. Findings and conclusions from the cognitive study

The table as a sign system integrates graphical form with the reduction of space and text and allows for establishing relations among the objects being dealt with, activating classification schemes and symbolic systems. This entails a more complex way of thinking, complexity that has repercussions in the cognitive processes involved in its use and in difficulties in learning tables.

The path on which we have embarked in this research is eminently cognitive, given that we investigate the possible schemes that children use in producing tables and the cognitive demands that the teacher orchestrates to put invariant operators or new *schemes* in action (Studies 1 and 2, respectively). The teacher's demands promote progress in the learner, in order to advance in the complexity of the situation and the acquisition of new meanings. At the same time, considering *statistical literacy* and specifying fundamental statistical ideas allows us to distinguish the statistical thinking that the SRLE teaching model proposes to develop.

As we consider that the socialization process of table meanings begins before entry into the school system, we propose an open ended data analysis learning situation. The students' performance in resolving the situation was diverse, due possibly to their previous dominance of some semiotic representations and similar situations which they had faced previously.

The student productions included simple lists without counting, lists with counting, simple pseudo tables with icons instead of words and repetition of icons as an expression of counting, pseudo tables with categories and counting but without spatial ordering in rows or columns, graphical tables with icons repeated in columns, among other representations.

The empirical data allows us to characterize progressivity in the conceptualization of frequency tables. We identified 3 stages, which include: list schemes (iconic, textual, without counting), pseudo tables, and tables with counting and marginal totals. A recurrent factor in these student productions is repetition, with or without order, of the elements of a class; this makes us suppose that ordering the repeated elements would facilitate quantitative visual estimation and, as a consequence, also facilitate counting and efficiency in visual search processes.

For the group of grade 3 students faced with the data analysis situation, and considering the discrete character of the frequency table, the *schemes* activated were association, differentiation, and perception of order and quantity. As such, the theorems in action present in all the productions are:

(1) "An equivalence relation determines a partition and a partition defines an equivalence relation" related to the processes of association and differentiation; and

(2) "Card $(A \cup B)$ = Card (A) + Card (B) – Card $(A \cap B)$" related to order and quantity.

Vergnaud affirms that knowledge is action or it is not knowledge. We recognize in his theory that knowledge manifests itself in two ways, operative and predictive, and it is the operative form of knowledge that lets us see knowledge in action. While we cannot affirm that children's process of conceptualization of the table passes through each of the stages described, we can presume that competencies in situations of ordering and organizing data influence the progress and recovery of knowledge produced in the process of conceptualization of tables. From the perspective of the transnumerative ability expressed in the student productions in the lesson documented in this dissertation (and also in four other implementations of the same lesson, one in grade 1 and three in grade 3, mentioned briefly above) it was possible to confirm the four phases of transnumeration identified by Rubick (2004), that is, recognition of the data's message, choice of representation, transformation of the data, and representation of the transformed data. In the lesson reported, in particular, we see the transnumerative techniques of grouping, sub-set selection, change of variable type, frequency calculation, graphing/tabulation, and other calculations.

The table as an external symbol is recognized visually and symbolically through its properties and spatial relations. The analysis carried out leads us to hypothesize that children could progressively develop different schemas for the table concept, depending on the learning experiences they have encountered initially, for example, the construction of iconic lists without counting and with evident repetition of elements and spot and sequential readings associated with these lists. Confronting children with this type of experience would allow them to appreciate the advantages of frequency tables, with or without demarcating lines, in which the differentiation and order is evident, in order to activate processes of searching or comparing through sequential readings (in rows and columns).

More precisely, recognizing the theorems in action that support the conceptualization of tables provides knowledge that allows us to define learning paths that potentiate continuities and let us confront ruptures in the concept. The analysis carried out suggests that in order to achieve the conceptualization of tables, one needs:

(1) To consolidate capacities of association and differentiation, of ordering, and of quantity;

(2) To have experiences with the concept of lists, as we consider them the basic unit of tables, and their vertical or horizontal dispositions allow different spatial readings;

(3) To make explicit and value the communicative components created in reduced form (headings with the names of the variable, its categories, and classes);

(4) To consider various situations that provoke the necessity of creating tables, allowing for different systems of representation (iconic, written, and numerical); and

(5) To allow for creating tables with and without physical demarcations, considering that mental segmentations facilitate point readings (cells), sequential readings (rows -horizontally- and/or columns -vertically-), or global readings (the entire table or parts of the table).

Studies 3 and 4 were also cognitive in nature and sought to differentiate the processes of understanding tables to determine the different cognitive demands associated with them. The focus was on developing a taxonomy of table understanding according to the table structure and the role of the subject according to the purpose of the associated task, and on studying the degree of concordance of the taxonomy to verify its functioning.

The principal result obtained is the taxonomy of table understanding, based on the identification of the reading flows associated with different tasks using tables and the classification of the reading processes as a function of tables' structural components. In summary, the taxonomy of table understanding integrates the physical components of tables (rows, columns, cells) with the reading of data -arranged in lists or cells in the body of data- and the variable categories -found in the margin-.

The taxonomy of table understanding developed includes types of reading associated with a local area, a part of the table, or the entire table. Specifically, for level 4, extensive reading of the table involves a critical reading of the table and its repercussions; for level 3, intensive reading implies a global reading of the table; for level 2, sequential reading means reading a list of data that can do without some cells and/or headings; and level 1 implies a spot reading of cells or lists that does without the headings entirely.

From a cognitive perspective, tables belong to a relational hierarchy superior to lists, as they establish spot, sequential, and reticulated binary relations, and can also establish multiple relations. To establish levels of complexity in dealing with tables, we paid attention to reading types and subjects' roles according to the tasks' purposes. In this study we identify the roles that activate internal processes of searching, interpretation, and evaluation, and external processes of building, completing, and recording.

Following the findings of Gabucio et al. (2010), the taxonomy we develop adapts itself to the physical structure of tables. The contribution of this research was to introduce the focus on the positioning of data, which regulates the reading of the data, and also on the purpose of the associated task. Based on the premise that tables and graphs respond to different needs, one with respect to specificity and the other with respect to tendencies, the taxonomy developed here is different than the graph taxonomy. However, to promote mental association, we link the taxonomic levels identified for table understanding to Curcio (1989) and Aoyama's (2007) taxonomic categories for graphs, recognizing the theoretical differences on which they were built.

The practical purpose of Studies 3 and 4 was to conceive an instrument that allows us to anticipate the level of difficulty of tasks using tables and recognize that tables possess a physical structure of location and semantic content that lets us work with them.

Study 4 sought to determine whether the levels of graph understanding are the same as the proposed levels of table understanding. Through statistical analyses, it was possible to determine that the taxonomy of graph understanding is not similar to the proposed taxonomy of table understanding. As such, the study contributes a taxonomy of table understanding that uses concepts, structure, and language specific to the table format.

4. Findings and conclusions from the didactic study

Some findings from the literature review

Archaeological studies in an ancient Babylonian city let us recognize tables as a knowledge object that came into existence when scribes -in their role as agents of the teaching system- became aware of tables and introduced them to the system of objects to be taught -or objects to be learned- due to their usefulness for the economy of the didactic system that these schools faced.

In order to form didactic knowledge about tables, we studied their status in the current Chilean primary school mathematical education program of study in the data and probability theme and in primary school mathematical education programs of study from some OECD countries[30], and their role in items on international tests such as the TIMSS. Their status was also studied taking into account the institutions and activities that use them in informatics and statistics.

A contribution to teaching tables at the school level is the generic table model, which concisely shows the structure of physical location with the structure of the statistical contents (see Figure 68). Recognizing what is meant by “table” and the agreed upon language for table structure appears to be necessary knowledge for understanding tables; as such, these constitute basic elements that contribute to learning and teaching tables.

[30] Curricula of countries -some of them successful in international tests- that explicitly include the table object.

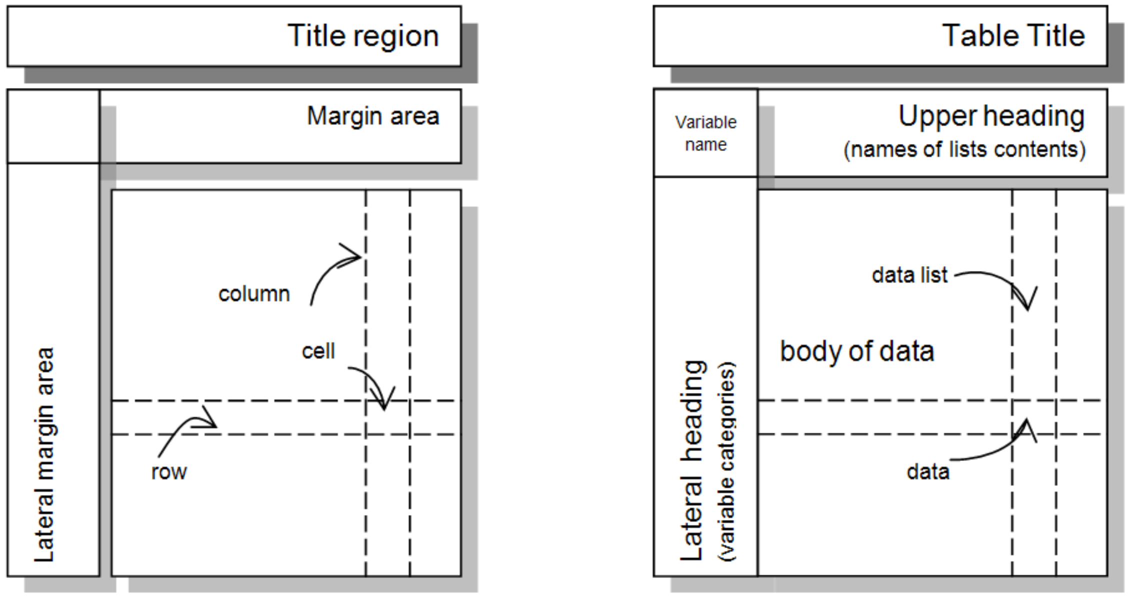


Figure 67. Generic table models according to physical structure and content.

Another contribution obtained in the review of the statistical literature was the recommendations for making tables with a communicative aim. As tables display discrete values in discrete categories of rows and columns, understanding tables requires some strategies for reading the numbers they contain and comparing between columns, not many elements -around 20, but not more than 50-, limited number of digits, decimal alignment, parentheses reduction, order with meaning, inclusion of statistical measures, good use of space, and moderation in design. We consider that, as tables possess all the data, the analytical focus is on the local reading and interpretation of the numbers, principally in comparisons among them and, eventually, in the search for global tendencies in the data.

Some findings from the curriculum

The Chilean curriculum regarding data and probabilities seems distant from the ideas of exploratory data analysis (Tukey, 1977) in statistical education, which respond to a general movement that promotes and values the use of representations as an analysis tool and not only as a means of communication.

Also, the present curriculum presents critical absences, such as the concept of variable, dealing with scales and coordinates, the change from one representation to another, and prediction. In the analysis of curricular activities we found a lack of cognitive activities such as "build-visualize-communicate" that are important in mathematics, science, and the development of scholarly scientific research, and are relevant to students' performance on international tests.

The revision of the curricula of England and Brazil allowed us to find in their proposals a sequence from lists to tables -coincident with the epistemological finding-, as they consider lists as their basic unit. Nisbet (2003) describes a prior study that examined the representations generated by 114 teacher education students. These students created 11 different types of representations ranging from lists (ungrouped and grouped) to various types of tables, pictographs, line plots and bar graphs. Again, and from a didactic perspective, our study suggests that these representations emerge in data analysis and we give evidence that, independent of age, the path to tables must pass through lists, and that lists are part of the configuration and conceptualization of tables.

Finally, studying the TIMSS items allowed us to outline the statistical situations using tables and the levels of cognitive demand. The analysis of the items from the international test show in the majority of the items an arithmetic reduction of statistics with activities centered on completing with numerical equivalencies or other arithmetic operations of an immediate and decontextualized nature.

The Study

Study 2 was didactic in nature and made an analysis of the cognitive demands made by the teacher in the lesson. Specifically, it sought to characterize teaching that favors statistical reasoning by the students through data analysis and the use of tables, and the development of reasoning with high level cognitive demands.

The proposed learning situation for the emergence of frequency tables as a cultural and meaningful construction yielded productions that were more or less useful in the task of classifying and representing the data. What is interesting about the lesson implemented is that all of the strategies were respected -as a lesson norm- as they responded to the problem presented, and the students themselves chose the best strategy and provided arguments for it.

The lesson was planned with a focus on student-centered teaching; to do so, the Statistical Reasoning and Learning Environment (SRLE) model was used, which favors the development of deep and meaningful understanding of statistics and promotes high cognitive demands in order to reason statistically "doing statistics". Through observing the lesson implemented and its learning environment guided by the lesson plan, we see that it achieved that the students reflected on what they were doing. The teacher encouraged the communication of the ideas produced in the classroom through discussion and argumentation, and both in the productions and the arguments the level of responses to the initial problem and the level of understanding of the data analysis concepts that the students achieved can be evaluated.

5. Contributions

One of the contributions of this research is to provide evidence for the development of an object, the table. We present tables as a *proto-mathematical* object, next as useful for studying other mathematical objects, hence a *para-mathematical* object, and more recently in their status as a *mathematical object* to be studied in its own right.

Likewise, the taxonomy of table understanding is a contribution of this research, as it was the product of various antecedents and systematic analyses, such as the roles of subjects faced with tables, tables' reading flows, the analysis of the type of use given to tables according to historical milestones, and the analysis of items from international tests. The taxonomy's analysis involved aspects of reliability and internal validity, of construct and content, and a statistical analysis of the difference between this taxonomy and a graph taxonomy.

A didactic contribution from the epistemological and cognitive studies is the recognition of the list as an antecedent of the table and of the list scheme as a solution strategy for organizing data in diverse age groups. Identifying this transition contributes to the understanding of tables regarding their conceptualization. The knowledge that lists are elemental constituents of tables, and as such should have a place in the curriculum, is an issue dealt with years ago, at least by England and Brazil.

A contribution related to knowledge about tasks related to tables was establishing the action of reading tables as a crosscutting and basic action for any task, and identifying two groups of actions relative to tables: record, search, and complete; and build, interpret, and evaluate.

Another contribution was the lesson plan, designed and perfected in the lesson study modality, that responds to a recent teaching model in statistical education, the statistical reasoning learning environment (SRLE), which favors deep and significant understanding of statistics and promotes high cognitive demands in the learners.

6. Future Prospects

6.1. From the theoretical framework

Based on the analysis of the table items on the TIMSS, investigating the arithmetization of statistics appears as a possible current didactic phenomenon to be researched, an issue that it would be good to enter into greater depth in, as in school culture there exists a simplistic form of interpreting the study of statistics as a succession of arithmetic procedures. This leads to considering in future studies the epistemology of statistical tables, paying attention to statistics' historical periods, understanding statistics as a recent discipline, in the style of mathematical education, in which relevant milestones regarding the numerical sequence are observed: arithmetical, set theoretical, and post set theoretical.

In this vein, knowledge of the process of transnumeration in statistics as development of transnumerative reasoning would allow for disseminating a statistical education that has its own distinctive scholastic goals, among these, cyclically investigating and searching for a better understanding of the problems through analysis of real data through representations.

The diversity of the productions of lists and tables for organizing data merits continuing to delve into the age group, the specificity of the *schemes*, and the transnumerative ability in play. Future studies should articulate the concept of transnumeration with Duval's (1999) concept of a record of semiotic representation for the coordination of semiotic systems, considering the development of Peircean abductive reasoning.

More studies are required for the table conceptual field, such as the identification of data analysis situations related to tables in schools and their classification, both relational and hierarchical, which specify the utility of a particular representation and under what conditions and in what instance this can be replaced by another.

Studies are also needed on students' difficulties in articulating invariant operators and in their clarity in the progressive mastery of situations in this conceptual field.

Some interesting theoretical models different to those assumed in this dissertation, and which could also be used in the issue that we have addressed are the instrumental approach as in the works of Rabardel (1995) and Trouche (2005), and Sensevy's (2007) joint action theory in didactics (JATD).

The implementation of the "snacks" lesson plan and the stability of the performance of the students in the lesson, allow for carrying out a study of the learning situation, for example, of its reproducibility, of its techniques (knowing how to) for analyzing *conceptualization* and the *techniques* used to carry out *tasks* for a certain subset of *schemes*.

The teachers' experience in carrying out a lesson study for a primary school data analysis learning situation based on the SRLE model of statistical education in our classrooms, invites us to develop research on the changes in teachers' beliefs and knowledge that came out of the learning and teaching environment of the lesson study and the SRLE, in order to evaluate its validity as an instance of teacher professional development. More research is necessary in statistical education to provide current knowledge about how adult subjects, learners and teachers, develop statistical concepts and reasoning.

It is necessary to implement the learning about tables in statistics in the curriculum, as the familiarity of tables in everyday life has positioned them as a somewhat transparent tool, which neither teachers nor students address in class, and this weakens the development of statistical literacy. This supports Friel, Curcio, and Bright's (2001), assertion that, in exploring representations of data, tables should be considered the most conscious form for students, given that they are used as tools for organizing data as well as for displaying it visually. They also observe that tables can be a bridge for transitioning from the representation of raw data to summary measures of data.

The open ended learning situation demanded the creation of a data representation. This research only investigates the conceptualization of tables in which the task requires their creation. It is necessary to investigate how this conceptualization progresses with the presentation of tables already made, and to look into tasks using tables that -using our terminology of task purpose- demand recording, searching, completing, interpreting, and evaluating.

6.2 From the results of the studies

This research advocates creating tables, both for ordering and classifying data in groups, considering the creative process as more cognitively demanding than others. Future research should study the processes involved in interpreting tables, so that learners develop the ability to relate what has been previously united, that is, reunite based on partition, establishing correspondence in what is explicitly shown to account for the implicit connections.

Some lists without counting and without repetition revealed the loss of information that entailed the loss of characteristics of the data, an issue that effects visual quantitative estimation and, consequently, makes counting and search efficiency difficult. Lists are made up of spatially organized discrete information. Given the apparent similarity of the two formats (lists and tables) we study the transformations that allow us to create a table based on a list. The step from a list's enumeration to a table implies identifying diverse underlying variables in the list, whose values are organized in two or more dimensions. Future studies should investigate lists in the first years of school with textual or iconic expressions and jointly taking into account the processes of ordering and classification.

A finding that also merits research and that cuts across age groups is the homogeneity of the productions of the children in grades 1 and 3, without prior instruction. Given this similarity, the conjecture emerges that grade level (or age) does not influence the sophistication of “natural” strategies for organizing data, and as such a teaching sequence is needed, as specified in our proposal, so children can master tables as an object beginning in the first years of school.

It is necessary to continue studying frequency table learning proposals in school statistics, as in both the exploratory study and the empirical study we found that, without prior learning of these tables, their emergence is not innate in learners. Only one girl in grade 1 (*n*=38) showed a pseudo frequency table in the exploratory study, and only two girls in grade 3 (*n*=80) showed a frequency table. Nisbet et al. (2003) also reports that the process of reorganizing numerical data in frequencies is not an intuitive process in children in grades 1 and 3. Additionally, Lehrer and Schauble (2000) conclude that, in the process of classifying, it is not easy for learners to use the criteria of recognizing, developing, and implementing.

A common characteristic of the two studies regarding levels of table understanding was the use of the taxonomy on some of the same items based on which it was created. That is, some of the characteristics of each level of table understanding were reconciled with the performance levels on the TIMSS (2003, 2007, and 2011), and the reading flows and the roles of the subject faced with the task were based on these items. While on one hand, the use of the same battery of items gives it coherency, on the other hand, it restricts its range of application. As such, future studies should provide population validity, that is, external validity that describes to what degree the results obtained can be extrapolated from the sample used to an entire population.

We have tried to contribute knowledge about tables regarding data in statistics, both as a representation in themselves and as regards the complexity of converting them to other representations. We also hope to have contributed to making tables more visible as something to be learned and taught in school, showing them as a tool and as an object and identifying the different roles that subjects faced with tables assume.

We finalize by emphasizing that the knowledge about tables collected in this dissertation and the future prospects outlined should contribute to tables being considered a teaching and learning object in the first years of school, so that students gradually master them as a cognitive tool, and so that teachers, curriculum developers, and textbook authors address their configuration, properties, and operations, and their distinct cognitive demands.

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ANEXO III.1

Some important highlights of tables and their possible functionalities

(M= almacenamiento de memoria   C=herramienta de cálculo   A= herramienta de análisis)

Años	Creador (es)	Desarrollo	Funcionalidad
4500 a. C.	Babilonios	Censo de población a escala nacional	M
2600 a. C.	Babilonios	Primera tabla matemática de datos en Shuruppag, tiene tres columnas con diez filas: con lista de medidas de longitud y otra para el área	M
200 a.C.	Egipcios	Tablets de Akhmim, o de El Cairo, contienen listas y algunos problemas numéricos de equivalencia de medidas de capacidad	M
1800 a.C.	Babilonios	Tabla conocida como Plimpton 322, con 4 columnas y 15 filas de números	M
366-335 a.C.	desconocido	Tabla de Peutinger: mapa de ruta del conjunto del mundo romano con distancias	M
100 a.C.	Griegos	Tabla de Keskintos, una tabla con cálculos de movimientos de planetas	M-A
2 d.C.	Chinos	Primer registro completo del número de familias y población del Imperio	M
100 d.C.	Egipcios	Tablets Stobart, cuatro tablillas con los registros anuales de los movimientos de cinco planetas	M-A
150 d.C.	Ptolomeo	Primera tabla trigonométrica con datos y cálculos en 360 filas y tres columnas	M-A
267 o 276 d.C.	desconocido	Tablas de barro de Astorga: contienen vías de caminos y distancias en el noroeste de la península ibérica.	M
1252	Astrónomos islámicos	Tablas Alfonsinas son tablas astronómicas realizadas por iniciativa de Alfonso X el Sabio para actualizar las Tablas de Toledo	M-A
1494	Luca Pacioli	Primer sistema de registro contable integral, organizado por cuentas y todas ellas equilibradas.	M-C
1551	Erasmus Reinhold	Tablas Prusianas son tablas astronómicas que trataron de actualizar a las Alfonsinas y que diseminaron los métodos de cálculo de Copérnico	M-C-A
1600	Alemania	Tablas de datos empíricos, publicaciones de tablas de números comienzan a aparecer. "Die Tabellen-Statistik", como una rama de la estadística dedicada a la descripción numérica de los hechos	M-A

1614	John Napier	Tablas de senos y cosenos con sus respectivos logaritmos	C-A
1627	Tycho Brahe – J. Kepler	Tablas Rudolfinas tablas astronómicas que contienen las posiciones de alrededor de mil estrellas medidas por Brahe, 400 más que Ptolomeo	M-A
1660	John Graunt	Primera tabla de vida (life table) con las cifras de mortalidad de Londres	A
1671	François Barrême	Libro de tablas matemáticas prácticas, destinadas a evitar cálculos engorrosos en el ámbito monetario (barrème devino en baremo)	C
1671	Jan de Witt	Tablas de mortalidad para determinar científicamente el precio de compra de rentas vitalicias	C-A
1693	Edmond Halley	Primeras tablas de mortalidad reales, que contienen las edades de muerte de una muestra de individuos bajo condiciones estables.	C-A
1766	Tobias Mayer	Tablas de distancias lunares para determinar con precisión las posiciones en el suelo, usando una fórmula de Euler	C-A
1767	Nevil Maskelyne	Tablas de distancias lunares dando la distancia de la Luna desde el Sol y las estrellas nuevas. Luego se convertiría en el almanaque estándar para los navegantes de todo el mundo, ya que llevó a la adopción internacional de Tiempo Medio de Greenwich como un estándar internacional.	C-A
1779	Johann Lambert	Primera visualización semigráfica que combina formatos tabulares con gráficos	A
1797	Louis Pouchet	Primer intento sistemático de construcción de tablas gráficas de doble entrada.	A
1835	Adolfo Quetelet	Tablas de frecuencia sobre datos de población en las cuales basa su concepto de “hombre promedio” (pero sin considerar las características marginales)	A
1844	Charles Minard	Tabla-gráfica para mostrar el tráfico comercial	A
1846	Augustus de Morgan	Tabla con columnas que representaban entradas de dinero (debito) y las filas salidas de dinero (crédito) de tal manera que la primera fila y la primera columna correspondían entre sí	C-A
1856	Florence Nightingale	Tablas con datos de estadística para tomar decisiones respecto a epidemiología e higiene sanitaria	A
1860	Ramón Picarte	Tabla de Logaritmos a 10 decimales (superior a la Tabla de Lalande	C-A
1869	Dmitri Mendeléyev	Tabla periódica de los elementos	A

1883	Ramón Picarte	Grandes Tablas de Logaritmos con doce decimales	C-A
1883	Touissant Loua	Primera tabla semi-gráfica que muestra una tabla de datos con niveles de sombreado	A
1893	Charles Sanders Peirce	Primera Tabla de verdad, o tabla de valores de verdad, que muestra el valor de verdad de una proposición compuesta	A
1898	Ladislaus Bortkiewicz	Tablas de clasificación cruzada de las muertes de soldados por patadas de caballos de la caballería prusiana, para argumentar la Ley de los Pequeños números (Poisson)	A
1904	Karl Pearson	A partir -y con- las tablas de contingencia desarrolla teoría para distribuciones de frecuencia multivariadas	A
1912	Bertrand Russell	Tabla de verdad, (su versión desde el atomismo lógico)	A
1921	Ludwig Wittgenstein	Tabla de verdad, o tabla de valores de verdad, (se tornan más populares, con el aspecto actual)	A
1979	Daniel Bricklin y Bob Frankston	VisiCalc, el primer programa de hoja de cálculo moderno que manipula datos numéricos y alfanuméricos dispuestos en forma de tablas compuestas por celdas	A
2004	Edward Tufte	Sparklines son diferentes y diminutos gráficos insertados en tablas con información condensada en menos espacio y más intuitiva "datos intensos, diseño simple, gráficos del tamaño del texto".	A

Although some functionalities of the tables presented hold more than one role, this table allows to conjecture about distinctive periods of uses of the table. Initially, mainly as a repository of memory; then, a period of mixed use, the table with the functionality of memory and calculation; then, the table with memory functionality and analysis; and a final period of the table used primarily for analysis functionality.