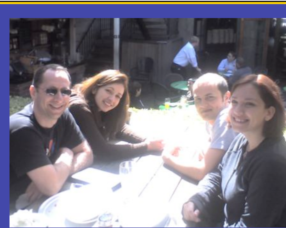

Bridging Theory:

Activities Designed to Support the Grounding of Outcome-Based Combinatorial Analysis in Event-Based Intuitive Judgment—A Case Study

Dor Abrahamson
University of California, Berkeley

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I'd like to use the opportunity of this presentation to share aspects of the research that are difficult to communicate in text -- the actual materials used in this study, and a video of real data.

Bridging Theory, Bridging Learning

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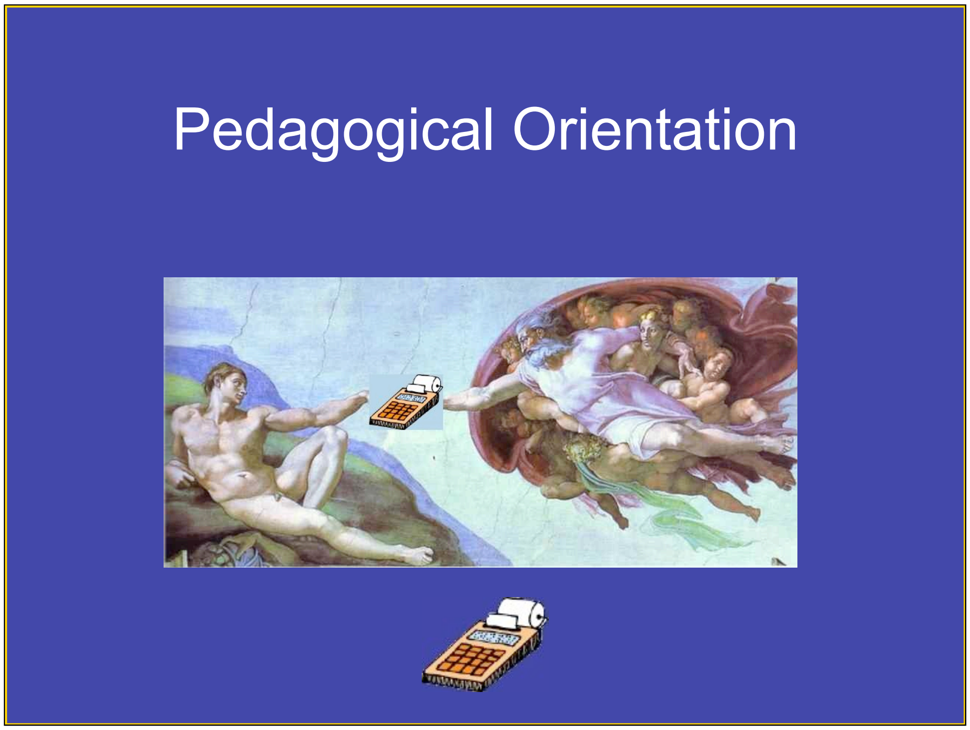
My talk is about two bridges, which I see as reflections of each other:

-a theoretical bridge between predominant interpretations of the nature of learning -- constructivism and socioculturalism

-and an educational-design bridge between students' intuitions and the calculus of probability

I'll be talking about the learning process of a student in Grade 6 (11.5 years old), and I will treat this learning process as a case study of these reflecting bridges between theory and design.



Well, here’s an international icon that I’m guessing most of you have seen before.

But here’s a twist on this picture. Let’s think of Adam as “the student” and God as “society.” Ok? I know that’s a bit odd, but bear with me. What this lets me do, is to caricature a few pedagogical frameworks, so as to situate my own orientation in this paper.

So let’s further say that the student -- that’s Adam, on the left -- has certain intuitions, and God has certain knowledge that the student is to learn.

I’ll use a calculator #

…to depict an artifact -- specifically a procedure -- that people can use without understanding. How is Adam going to learn to use the procedure?

So a caricature on traditional teaching would be #

…just “shove it in his face” and he’ll practice using it until he’s an expert -- what more might we wish for?

Then there’s a caricature on Radical Constructivists or perhaps Realistic Mathematics Education, who say, Well, we’ll create conditions so that Adam will build only on what he already knows and come all #

…the way to the knowledge. He’ll re-invent the calculator.

But many of us here are searching for something in between: guiding students to build meaning for mathematical procedures.

So today I’m asking a canonical question, Can we meet half way? Can we help Adam build on his intuitions even as he is learning to use a procedure?

“P(3H, 1T) > P(4H)” (HHHT, HHTH HTHH, THHH) <-> (HHHH)

“P(HHHT) > P(HHHH)” Revisited

- “*p*(HHHT) > *p*(HHHH)” = bias-prone heuristic
  - Kahneman & Tversky (1974)
- Replicated, analogously, with 8-month olds
  - Xu & Vashti (2008)
- Resilience to conceptual substitution
  - Saughnessy (1977)
- Legitimate reconstruction of problem
  - Borovcnik & Bentz (1991); Chernoff (2007)
- Misconceptions reconceived
  - Smith, diSessa, & Roschelle (1993)
- Enabling constraints; ecological adaptation
  - Gelman & Williams (1998), Gigerenzer (1998)

Naturally, I’ll be asking these questions in the context of probability content.

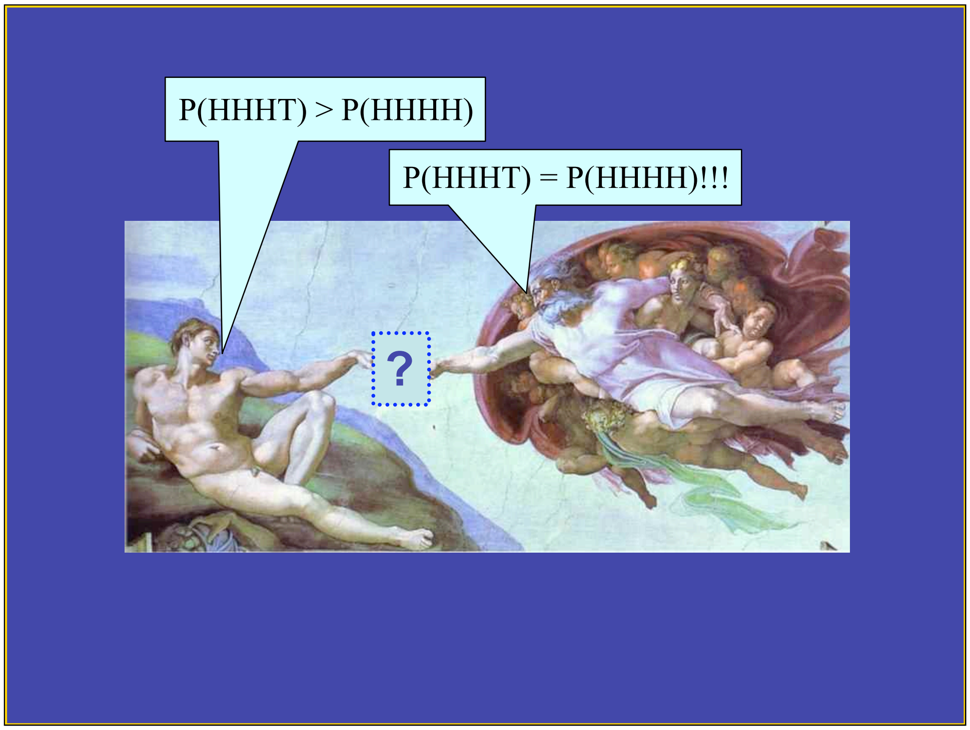
Perhaps the most robust finding from cognitive science that bears directly on the learning of probability comes from Kahneman & Tversky’s work. # As I’m sure you know, when they asked people to compare the likelihoods of these two outcomes from a sequence of coin flips, people thought that HHHT would occur more often than HHHH -- this, in contrast with probability theory, by which the two outcomes are equally likely (or ‘equiprobable’) for a fair coin. # …Recently, this finding has been replicated with very young children, of course using a different paradigm that included balls of two colors rather than these ‘H’ and ‘T’ symbols. So what are we to do with these mathematically incorrect judgments? Well, a very plausible idea is that we should do all we can to help kids get over this and move on. I mean, if kids thought that the capital of Mexico is Vienna, we’d probably tell them quite flatly that they are wrong, right? So why not do the same in this case? # Well, we have fine studies, such as Shaughnessy’s, that concluded that it is very difficult to overcome these intuitions toward developing what Fischbein calls a ‘secondary intuition.’ But there are additional interpretations of the Tversky & Kahneman finding, and therefore different implications for the pedagogy of probability. Namely, # …perhaps when people are asked to compare HHHT and HHHH - notwithstanding how clearly we state the problem -- …perhaps people are nevertheless interpreting the question differently? Moreover, perhaps people’s judgment demonstrates NOT a deficit but a strength? #

So we’re asking ourselves,

- *How* *are* people reconstructing the problem so that they arrive at the inference that HHHT occurs more often than HHHH?

-What might be the deep-structure cognitive element that gives rise to this construction of the problem?

If we knew the answers to these questions, we’d be in a position to explore the design of learning activities that *build* on students’ cognitive resources rather than attempt to eradicate them. So here’s a suggestion for how people are reconstructing the problem. People are ignoring the order of the independent outcomes and attending only to the number of T’s and H’s. Let’s mark this construction so. # Now, if people’s reconstruction of the problem was contradictory to acceptable mathematical knowledge, we’d be at odds as to how to proceed. But it so happens that # …this legitimate construction of the problem, that I’m suggesting students are entertaining, is in fact in agreement with mathematical knowledge. That is, there are more ways of getting 3H & 1T that there are ways of getting 4H, so -- when constructing the problem this way, it is actually true that HHHT occurs more often than HHHH. Ok, so where does that leave us? Specifically, if we go back to Adam for a moment, Can Adam use his intuitive judgments of likelihood to make sense of combinatorial analysis? More specifically, can Adam use his intuitive judgments of likelihood that appear to ignore the order of independent outcomes, ...to make sense of combinatorial analysis, a procedure that explicitly requires attending to order? What might a design look like that works WITH students’ intuition? And what can we learn from implementing such a design?



So, to sum up where we are just now, we've observed that human intuition #

Is not in accord with mathematical knowledge #

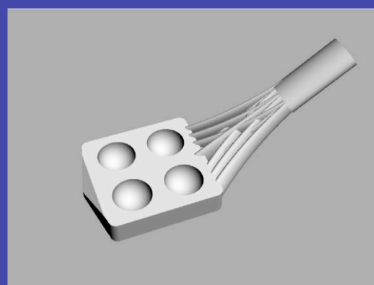
And yet, to the extent that we are committed to work WITH students' intuition, we are asking,

What mathematical situations #

…could support a teaching and learning of target content, that both embraces students' intuitions and enables the students to build on these intuitions toward the target knowledge?

Now, I'd just like to note why, I think, these questions emerged in this study. First, probability -- at least as it is proverbially perceived -- is counter intuitive, meaning that intuition and concepts drive contradictory inferences. I'm not the first person to say that this perception of probability as counter-intuitive is largely an artifact of how the key constructs and information of probability and statistics are represented (see the work of Uri Wilensky, Gerd Gigerenzer, or Cliff Konold). But secondly, in this particular design, as you shall see, we created situations in which intuitive and mathematical knowledge stand out in clear juxtaposition. This was a design principle -- we were hoping to foster a certain kind of cognitive conflict -- but it also meant that we had a very auspicious context for researching this juxtaposition.

The *4-Block Marbles Scooper*



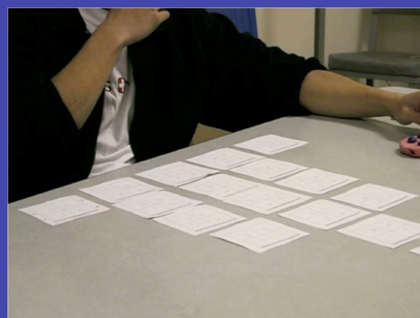
Empirical

Enter the marbles scooper. It’s basically an open urn from which you can draw ordered samples of exactly four marbles. It’s practically the same as flipping 4 coins simultaneously.

In building this random generator, I attempted to scaffold combinatorial analysis by introducing as an integral component of the sampling device and process a template for building the sample space. This is different from flipping coins, because the coins just fall all over the place and their spatial spread does not translate into the linear sequence of Heads and Tails that we use as a template for building the sample space.

We asked 28 kids from Grade 4, 5, and 6 -- that’s ages 9 to 11 -- what they think we’d get when we scoop. It turns out that these kids have a basically sound intuition for the expected distribution, -- by and large, students said that we’d get a scoop with 2g2b most often, scoops with 4g or 4b infrequently, and scoops with 3g or 3b somewhere in between. But when we asked the students to explain or support their estimates, they were lost for words.

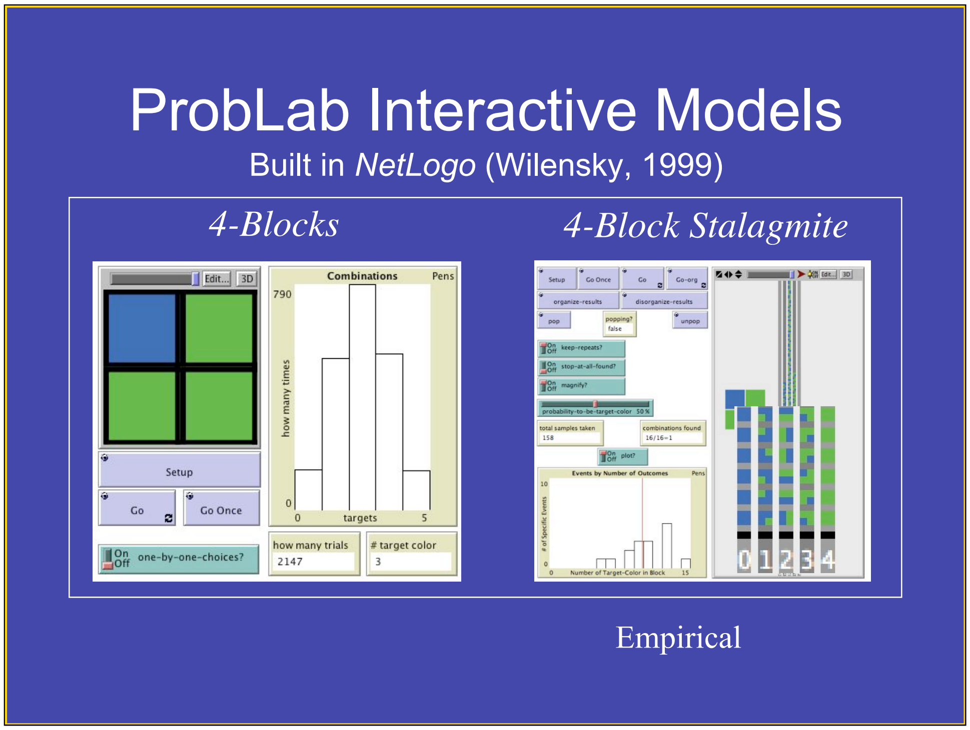
The *Combinations Tower*



Theoretical

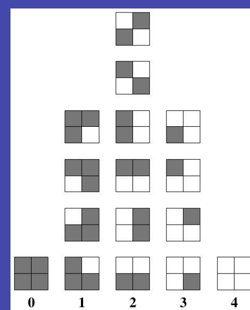
We then turned to another activity, what the Monty Python crew might call, And now for something completely different -- We had the participants build the sample space of the experiment. And once they built the sample space, we guided them to arrange the 16 cards in this shape, that we call the combinations tower.

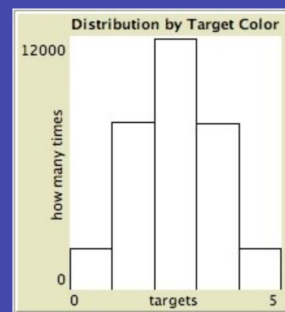
At this point, something curious happened -- These students, who did not know why they were engaged in this activity of coloring in all the possible outcomes, had an insight that the sample space actually has something to say about the mathematical situation. In fact, the students said that we'll get more scoops with 2g2b because there are more ways of getting such a scoop, as seen in the sample space. But what happened before and after that moment of insight is perhaps more interesting, as we shall soon see.



And there are computer based models built in NetLogo -- Uri Wilensky’s software for modeling and simulating science and social-science phenomena. This talk is not about the computer models, so I won’t say more about them. They’re all online on my lab’s server, and I’ll be more than glad to demonstrate them to anyone who’s interested.

Seeing Chance Learning Tools



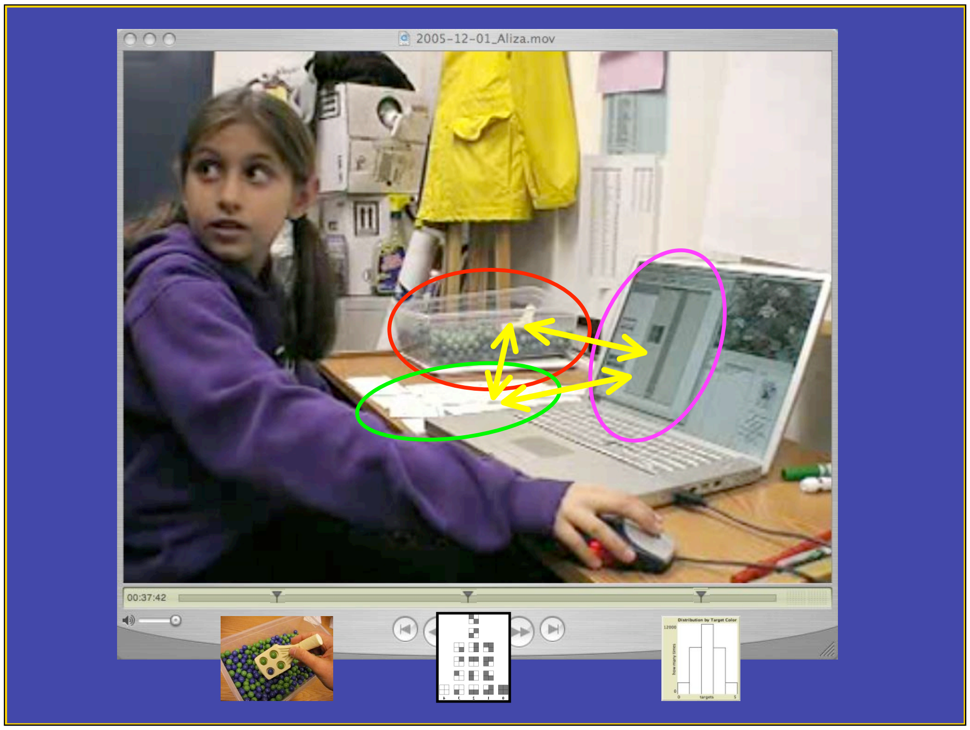


Empirical　　Theoretical　　Empirical

The 4-block, the thematic object of this design, is embodied in three different media.

Two of these media, the marbles box and the computer, embody the randomness generator -- the marbles box is the original experiment and the computer hosts an interactive simulation of this experiment. A set of cards that students construct constitutes the sample space of the random generator -- all the 16 unique configurations of green-blue on the 4-block marbles scooper -- and when these are sorted in groups according to the statistic 'number of green squares', we get what we call the Combinations Tower, which expresses the expected distribution of actual outcomes in empirical experiments with the random generator. That is, the more samples we scoop and record, the closer and closer the outcome distribution will converge on the 1-4-6-4-1 distribution. In probability theory that is called the Law of Large Numbers.

Students learn through coordinating the three artifacts.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Cognition, Media, and Interpersonal Resources

- Selective visual attention to ambiguous figures (Tsal & Colbert, 1985)
- Intuitive rules: More A - More B (Stavy & Tirosh, 1996)
- Conceptual-blending theory (Fauconnier & Turner, 2002; Hutchins, 2005)
- The medium shapes the message (Olson, 1994)
- Ontological Imperialism (Bamberger & diSessa, 2003)
- Semiotic means of objectification (Radford, 2003)
- Professional vision (Goodwin, 1994), disciplined perception (Stevens & Hall, 1998)
- Maxims of dialogue (Grice, 1989)

See the video with Li

•So here's Li, a 6th-grade student, who's just completed building the entire sample space and assembling it as a combinations tower. Now, I'm about to play a rather long bit of data here, so I need to give you both the background, so you have context to interpret Li's remarks, and some pointers, so you'll be looking out for some of the statements that he makes.

•When we first worked with the marbles box, Li, like all of the 28 participants but one, estimated that we'll get 2g2b as the most likely event, etc.

•I then asked him to use the cards to show me what we can get

•Li built only 5 objects -- 0g, 1g, 2g, 3g, and 4g. Now, the curious point is that the moment he had built those cards, he changed his prediction for the distribution, saying that the 5 events are equally likely. We discussed that point for a while, and then I asked him nevertheless to build the remaining 11 cards, for a total of 16 outcomes in the sample space. Once the combinations tower was completed, Li still insisted that there would be an equal distribution of outcomes among the 5 events. So he is still making this assertion that is counter to his initial intuition.

•This excerpt begins with me basically reiterating what Li has just said to me.

•At one point, as you will see, Li explicitly states that we only need to create 5 objects in order to show "what you can get." He actually says that the other 11 "don't' matter." That means he sees no value at all in accounting for the permutations -- he sees no evident reason to include all of these alternatives, when just one card for each event would do the job very well.

•The turning point is when Li sees the entire collection of cards as a means of supporting his earlier statements about the expected distribution.

•Li thus alternates between two ideas: (a) we only need 5 to show what we can get; and (b) but the entire 16 explain why we et what we get.

•So Li appropriates a mathematical procedure only when he recognizes that he can instrumentalize the procedure toward objectifying his presymbolic notion.

Conclusion

- The “P(HHHT) > P(HHHH)” intuition can and should be used in teaching probability.
  - Can: The design of materials, activities, and prompts created an opportunity for Li to begin to coordinate intuition and mathematics.
  - Should: The intuition gives meaning to the procedure of combinatorial analysis.
- Synthesizing intuition and mathematics (Schön, 1981) is laborious but possible -- it requires guidance, reflection, time.

For the most, I have been presenting data and attempting to explain them. But now I'd like to step back and separate the data and the theory.

That is, I see my paper as running on two planes that could be seen as reflecting each other --

•an empirical plane, in which I described Li's learning path through the activity sequence,

•and a theoretical plane, in which I offered explanations for this learning path from two theoretical perspectives.

More specifically # On the empirical plane, we have been looking at a case study of a student who apparently uses perceptual judgments # …in order to build a *sense* of understanding of a mathematical phenomenon, yet he then still needs to infer, perhaps using abductive reasoning, # …the implications of his heuristic insight for the mathematical properties of the objects he has created through the procedure he has just performed -- he still needs to sort out how the semiotic tools that were imposed on him (the 4-blocks) are aligned with his intuitions.[ft] On the theoretical plane, I will propose that this student's struggle roughly maps onto a tension between two complementary theories of education #

-- constructivism #

and socio-cultural theory, …as well as their respective implications for the design and facilitation of mathematical learning.

So the case study of Li constituted # …a background against which theories of learning are brought in relief.

I see this as work in progress, -- the reflection between the learning path and the theoretical models is very rough # …and the bridges at both the planes of design and theorizing are certainly not ready for all passengers to ford #…But it is in building such bridges, I believe, that we move toward a deeper and richer understanding of mathematical learning, and -- so doing -- improve our capacity to author effective learning environments that are truly 'realistic'-- that is, they are both respectful of students' cognitive resources from their everyday phenomenal reality and # …acknowledge the need to realistically meet the curricular objectives of procedural fluency. …so I look forward to your feedback on this talk and on the proceedings paper.

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