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PITFALLS AND SURPRISES IN THE TEACHING OF PROBABILITY

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“What is the probability of picking one red and one yellow candy from a sack containing two red and two yellow candies?” On the basis of this simple problem we expound three different ways of thinking about the teaching of probability. 1) Experimentally: We pick two candies twenty times and count how many times we get one of each color. 2) Combinatorically: We can see in how many ways the four candies can form groups of two, and how many of these groups contain one candy of each color. 3) Theoretically: We think probabilistically.

Each of these approaches may give rise to much discussion and communication. It is also interesting to see how these methods may interact and supplement each other. In a video, I will show students teaching in a classroom and how the communication is directed. I raise the question: “What does a student/teacher need to carry out her teaching involving all three approaches?”

INTRODUCTION

A teacher needs to know that in general there are several ways to attack a problem and that she needs to know more than one method to solve a problem in probability. A good teacher also needs to be open to the possibility that some students may think quite differently from them self. This implies that the teacher must be able to communicate with them, listen to their arguments and try to understand their reasoning. Obviously, this requires that the teacher has a solid understanding of the subject she is teaching. Difficulties in teaching probability is discussed by many, for instance Batanero, (2004).

THE CANDY PROBLEM

The problem, as presented here, is from a textbook for 7th grade in elementary school. (Figure 1.)

Hege has two red and two yellow candies in a sack. Mia picks two candies (without looking).

a) How many combinations are there?
b) What is the probability that Mia gets one red and one yellow candy?

Comment: In this problem question a) immediately directs the student towards method 2, a combinatory approach.

Figure 1: Two red and two yellow candies.

In a workshop for teachers we asked them to answer question b) What is the probability that Mia gets one red and one yellow candy?

Here are some answers:

1) RR, YY, RY give $\frac{1}{3}$ probability
2) The probability to pick a red first is 0.5, but then you have only three candies left …
3) red–red

   red – yellow 2 of 4 $\frac{2}{4}=\frac{1}{2}$

   yellow – yellow
   yellow – red
4) the probability to first pick a yellow and then a red is $\frac{1}{2}\cdot\frac{2}{3}=\frac{2}{6}=\frac{1}{3}$

   same with red – yellow
5) $\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}=50\%$
6) y_1y_2 y_1r_1 y_1r_2

   y_2y_1 y_2r_1 y_2r_2 $\frac{8}{12}=\frac{2}{3}$

   r_1y_1 r_1y_2 r_1r_2
   r_2y_1 r_2y_2 r_2r_1
7) Table

	Y_1	Y_2	R_1	R_2
Y	YY	YY	RY	RY
Y	YY	YY	RY	RY
R	YR	YR	RR	RR
R	YR	YR	RR	RR

8) Experiment with counting:
   16 of 25 was one yellow and one red

2y	IIIII
2r	IIII
yr	IIIII IIIII IIIII I

The eight different ways show all three approaches to the problem.

A VIDEO WITH STUDENTS AND THEACHER STUDENTS

A video shows students in a classroom working with a similar problem: They are given a jar containing two blue and two pink marbles. They are asked to pick two marbles randomly. What can happen? What is most likely? Investigate.

This is of course the same mathematical problem as the candy problem, but it phrased more openly, inviting the students both to think and to investigate. The students found that three things could happen: They could get two blue marbles, they could get two pink marbles or they could end up with one of each colour. By investigation they discovered that the last possibility was most likely to occur. They were surprised to see how often they got two different colours, but did not take note of the frequency of this outcome. (Knudtzon, 2010).

TWO AT THE TIME OR ONE AFTER ANOTHER?

A 13 years old boy worked with a problem: If you have four cards, two red and two black, and you take two randomly, what is the probability of taking two cards with different colours? He said:

> If I draw both cards at the same time the probability is ½, but if I draw one card first it is 2/3 possibility that the next is different from the first, because then there are three left to choose from. But if I have decided which card to take next when I draw the first, then it is the same as if I take both at the same time and the probability is ½.

In the problems mentioned here the question always arises if it matters whether we take two (marbles, candies or cards) at the same time, or take one after another. Many students, teacher students and teachers as well will argue that there is a difference. (One answer is to try, to experiment, to use one method twenty times and the other twenty times and see if there is a difference.)

Sometimes I try the following:

1) Put your hand in the jar and grab two marbles without looking. Extract your hand and open your fist. Then you have either two blue, two pink or one marble of each colour in your hand.
2) Put both your hands in the jar at the same time and take one marble in each hand. Take your hands out an open them simultaneously. Will this be the same as in the first case? Yes, again you will have either two blue, two pink and one of each colour. But you may also note that you may have one of each colour in two different ways, you may have the pink marble in your left hand or you may have it in your right hand.
3) Put both hands in the jar at the same time and take one marble in each hand. Then open one hand first, for instance your left, then open the other hand. You end up with two blue, two pink or pink in your left hand, blue in your right hand or conversely.

A moment's contemplation will convince most students (if not all teachers) that the probability of each outcome is not affected by which of these three procedures you use. It does not matter if you use one or two hands (or any other instrument) to pick out two marbles, and it does not matter if you open one hand after another, the marble is in the hand regardless and it does not change colour by waiting to be looked at.

There is another variation which is more interesting and is the cause of much confusion.

4) Use one hand to extract one marble from the jar and put in a saucer. Put your hand in again, and extract another marble from the jar and put it in the same saucer. The outcome will of course be the same as in the other cases, there will be two blue, two pink or one marble of each colour in the saucer. The probabilities of each outcome will also be the same as before, for the procedure is actually the same as described in 3).

I am fascinated by all the different views we encounter when we teach probability. That makes teaching probability challenging an interesting, but very difficult for teacher students.

CONCLUSION

What are the qualities or disadvantages of the various approaches? Here is Table 1, summarizing some of my experiences.

Experimentation, investigation	**Combinatory**	**Probabilistic**
Practical	Requires correct counting	Quickest way
Everybody can take part	A safe method	General, handles complex situations
Functions also in situations where there is no mathematical probability		Enhances thinking and understanding
Inaccurate for small numbers, makes it difficult to separate small differences	May get confused, problems with finding all possibilities	May be difficult
Takes much time	Can take much time	
Difficult to use in situations involving conditional probabi-lity		

Table 1: Qualities or disadvantages of the three approaches

I want to emphasize the usefulness of investigation and experimentation, especially in the beginning. Everybody may join in, but it is important to write down the results in an orderly manner. Different groups may get different results and gives good opportunities for discussions. However, the teacher need to know all three approaches, and how it is possible to make them interact in her teaching. Here Deborah Ball's notion of specialized content knowledge and teaching, play an important role in teachers' becoming explicit on their teaching. (Ball & all, 2008)

References

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