What's Typical? Children's and Teachers' Ideas About Average

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1. Introduction

In sets of data, from the simplest to the most complex, one of the essential problems is to reduce a large, unmanageable, and disordered collection of information to summary representations that capture the essence of the data. Even in their first data collection experiences, young children begin to move from focussing on individual pieces of data ("I like chocolate ice cream best") to capturing the essence of a larger amount of data in some manageable form: *most* of the class likes chocolate ice cream best, *few* students like strawberry. As soon as the need arises to describe a set of data in a more succinct way, the need for descriptive statistics arises: What is typical of these data? How can we capture their distribution?

This study explored the guiding conceptions and misconceptions from which children and adults build their models of descriptive statistics. Unlike other empirical studies of children's conception of average, we focus on people's own constructions of the idea of average and explore the relationship between informal ideas about "typical", "representative", and "average" with formal definitions and algorithms learned in school.

Twenty-one children (seven fourth, seven sixth, and seven eighth graders) and eight teachers were interviewed, using a series of open-ended problems that examined their notions of average. Grade levels of the teachers were as follows: one fourth grade, one fifth grade, one sixth grade, two seventh grade, one eighth grade, and two were mathematics coordinators (one at the elementary and one at the middle school level). Clinical interviews provided a means of examining, extending, and probing participants' ideas in different contexts. Based on pilot work, we designed a series of seven problems (available from the authors), which included both *construction problems* and *interpretation problems*. Interpretation problems involve describing, summarising, comparing, and reasoning about given sets of data. For example, the Allowance Interpretation Problem asked students to imagine that their parents were willing to give them the "typical" allowance for students their age and that they had collected data from their classmates which appeared as in Figure 1; what allowance would they ask for, and what would their arguments be, based on these data?

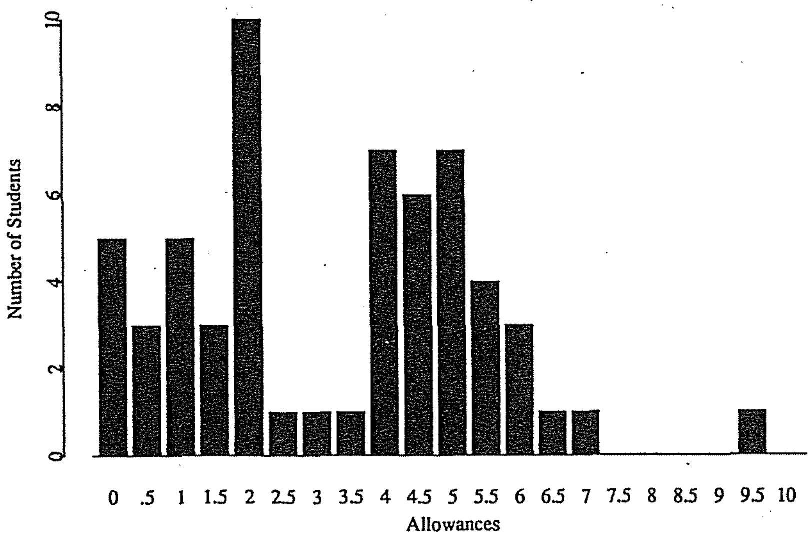

FIGURE 1
Allowances of 59 students

Construction problems call for participants to construct a set of data which might have resulted in a particular mean or other measure of average. These tasks resemble the way most of us encounter averages as part of the information we deal with daily; that is, given a reported average, usually with little other information, we must interpret what that average actually does and does not tell us about the world. The construction problems were particularly revealing of participants' underlying understanding of the relationship of an average to the data it represents. The construction problems included the Potato Chips Problem (if the typical price of nine different brands of potato chips is $1.38, what might the prices have been?) and the Allowance Construction Problem (if the average allowance of a group of students is $1.50, what might the data set look like?).

The analysis yielded four broad approaches to understanding average: (1) average as modal; (2) average as what's reasonable; (3) average as the midpoint; and (4) average as an algorithmic relationship.

2. Average as modal

The five children (three fourth graders, a sixth grader, and an eighth grader) and one adult in this group had a relatively easy time interpreting and constructing an average. The average was fairly consistently seen as the value that occurs most often. With this mode in mind, both interpretation and construction of an average were relatively easy. However, when they were asked not to use the average value as part of their distributions, people in this group encountered real difficulties. People in this group were not very flexible in their problem-solving approaches. Their alternative strategies for solving a problem were quite similar to their initial strategies, and often involved only a minor adjustment to the frequency of data placed on the mode. While some people in this category could recite the algorithm for finding the average, they rarely attempted to use it. For example, sixth grade Maria, when asked to explain what average means in the context of a problem about M&M candies, reported that, "The only average I know is like when we add up our grades and then see how many grades you got, and that's how I get the average. I don't know how to do it with M&Ms."

However, people in this group did show glimmers of understanding about the place of the average in the data. When undertaking construction problems, they did not necessarily place all - or even most - of the data on the mode. Their choices about where the data should go were dictated first by placement on the mode, but sometimes were determined by how reasonable the values were, and occasionally by their overall sense of the distribution. The following explanation of potato chip pricing, from a fourth grader, is typical: "OK, first, not all chips are the same, as you just told me, but the lowest chips I ever saw was $1.30 myself, so since the typical price is $1.38, I just put most of them at $1.38, just to make it typical, and highered the prices on a couple of them, just to make it realistic."

At first glance, it appears that many of the children in this category have simply developed a very literal meaning for the word "typical". Could it be a matter of semantics? It is certainly appropriate to equate typical with the mode. But the researchers spent a good deal of time probing for additional, deeper understandings of "typical", average, or representative. And despite the different contexts and multitude of requests, those in this category could not be prompted to construct other meanings of "typical".

3. Average as reasonable

The two fourth graders and two sixth graders in this category were "reasonable" in their thinking about average in two respects: first, they used values that were reasonable based on their own understanding of prices, allowances, and other phenomena. One clear meaning of average for these children is what they knew to be true in their own lives. But children in this group were reasonable in a second, critical way: they knew a fair amount about average as it relates to the distribution of data. Usually, they indicated that the average is roughly centred among the rest of the data. This was not a precise middle, but rather a sense that high values must be countered by low values when thinking about average. Children in this group were interested in the problems and saw them as very intertwined with real-life problems. They seemed to care about the problems, the data, and how the researchers got the data.

Initially, the reasoning of children in this group looked fairly egocentric and undeveloped. But there is an important element of reasoning in this category that is simply not apparent in "modal" reasoning. Children in the "reasonable" category have a concept of average as representative, a concept that is not yet connected to a precise mathematical construction of average, but is nonetheless a critical foundation for future development. However, this common-sense approach was not without its difficulties.

Children found the average fairly readily, but they thought about the average as an *approximation* rather than a number with a precise mathematical relationship to the data. Thus, on a weighted means problem, where average weights for two groups are given, these children did not think you could find an overall average. Even the two students who eventually derived the correct answer on the weighted means problem were unconvinced that their answers were exact. Both believed their answers were in the right ballpark (i.e. reasonable based on their knowledge and mathematically reasonable), but, as one student explained, "It's probably off because you don't know the exact weights of everybody."

4. Average as midpoint

Like children in the reasonable group, the six "midpoint" children (one fourth, one sixth, and four eighth graders) and several teachers in this category, were flexible and understood a good deal about the place of the average in distributions. They had a good intuitive sense of average, and understood what reasonable values for the average might be. These children and adults gravitated to the middle of the data as a way of characterising a data set. Most did not use a formal definition of median, but looked for or constructed a point around which the data appeared to be symmetrical. The adults and students in this category associated this point of symmetry with the mean of the data set, sometimes even suggesting that the median might be used as a shortcut for finding the mean.

Nearly everyone in this group dealt with the construction problems by placing an equal number of values above and below the mean, usually in a symmetrical manner. This approach resulted in a solution where the mean, median, and mode were identical. Most of the children in this group were quite facile at placing values in a symmetrical fashion around the mean. In some cases, this strategy appeared entirely visual. Sarah (eighth grade) constructed her graph by simultaneously placing tiles with her right and left hands, on opposite side of the mean. When finished with her distribution, she commented, "I think that might work, because I was just looking at it, and the spaces [on left and right side of the mean] are equal, so they have to be equal over here to be averaged at $1.50. I think that it would work just by looking at it."

All of the people in this group knew the algorithm for finding the mean and used it as a check on their more informal methods. Some of them quite obviously equated their median-like strategies as a short-cut for finding the mean. For example, Andrew, an eighth grader, explained that one could find the mean through the algorithmic method, or through his invented median method. He explained, "I think it would equal out to the same, but I think this [his median method] is a quicker way."

But what are the limitations of this view? First, children in this category frequently identified a middle which did not take into account important features of the data set. They might choose the middle of the range, rather than the middle of the data, as the average value. In this case, their average did fall between the extremes of the data values, but might not reflect the way in which the data were spread between those extremes. A more significant problem for those in this category was in dealing with skewed distributions and with more spread out distributions not obviously clustered around a well-defined visual centre. When we pushed for a solution on the Potato Chip

Problem which did not use the average of $1.38 as one of the values, people in the midpoint category were usually stymied. Because there were an odd number of prices, people in this category typically priced one bag at $1.38, four bags below this price and four above. Three of the six children said the problem could not be done without using $1.38, while two made a simple adjustment in one number (e.g. moving $1.38 to $1.37) and said the outcome would be close enough. Only one child - and a couple of teachers - came close to breaking the symmetry barrier with this new constraint.

The most obvious shortcoming of the midpoint strategy is its inability to deal in an effective mathematical manner with nonsymmetrical distributions. In many cases, "midpointers" simply ignored values which could not be balanced easily. When the assumptions of this view - that the data are symmetrically and closely grouped around a central point - do not hold, those for whom "midpoint" is the primary touchstone have difficulty both interpreting and constructing averages.

5. Average as an algorithmic relationship

The people in this group considered the average to be the value which results from adding up all the values in the data set and dividing by the number of values. Five of the 21 children and six of the eight teachers used this strategy to some extent. All of the children were sixth and eighth graders. The focus for those in this category is on the procedure for finding the mean, often with little attention paid to the reasonableness of the results or to the relationship between the mean and the data.

Those who were successful with the algorithm could jump in at different starting places: they could use the total to find the average, the average to construct a data set, and the data to determine both total and average. On the Potato Chip Problem, for example, the successful algorithm users figured out the total (9 bags × $1.38 = $12.42), then constructed prices that would add up to the total or broke the problem into submeans, constructing prices for three bags at a time with the sum of these values having a subtotal of $4.14.

Five people - three students and two teachers - in this category showed real confusion and became hopelessly tangled in numbers. These people understood that averaging involves finding a total and dividing by some number, and they were eager to use the algorithm when confronted with a problem. However, they overtrusted the algorithm, and, blinded by this trust, they were willing to give up what they knew about reality.

Those who had only a procedural understanding of average had great difficulty with the construction problems, since there is no way to use the algorithm in reverse to recapture the data set from the average. Some students created their own algorithms, procedures which resemble the submean strategy of successful algorithm users, but with values chosen idiosyncratically. For example, on the Potato Chip Problem, one child chose pairs of numbers where the "cents had to add to $.38" (e.g. $1.08 and $1.30). Her pairs consistently totalled $2.38, which she thought yielded an average of $1.38. Another child chose pairs that *differed* by 38 cents.

Those who used the algorithm blindly either did not have or could not access any sense of the relationship of the mean to the data. For example, one teacher for whom calculation was the arbiter throughout the interview, made a simple calculation error as she attempted to construct a distribution with a mean of $1.50. Believing that her mean was too high (it was not), she added more low values to her data until her distribution appeared as in Figure 2.

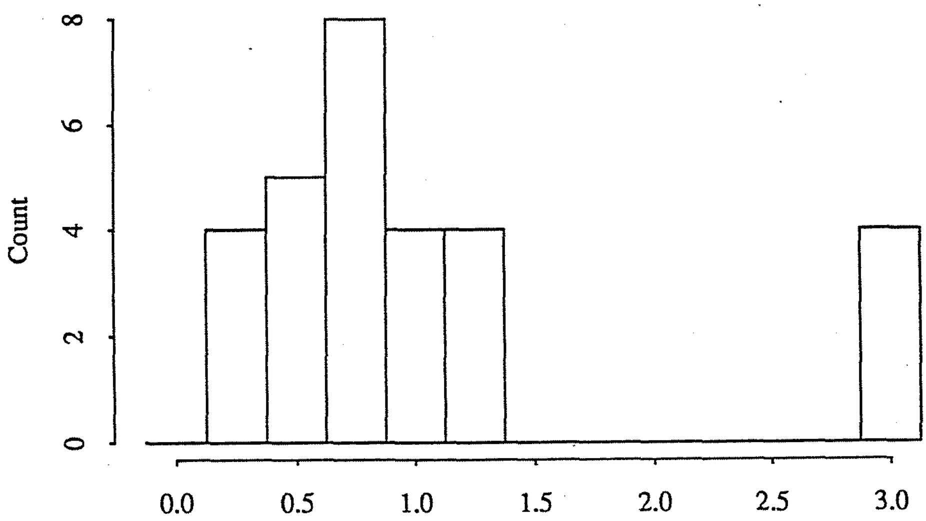

FIGURE 2
Evelyn's allowance distribution

She was surprised when her calculation now gave her an average below $1.50. In her dependence on the algorithm, she had virtually ignored the visual picture of the distribution in front of her. For Evelyn, the arithmetic relationship among mean, data, and total overrules the intuitive mathematical relationship of mean and data.

The constant calculation and recalculation that characterised people in the algorithm category is a clue that something important is missing in the understanding of average, even for those who used it successfully. It is as if a pianist were technically flawless, but lacked any feel for the music. If there were a feel for the data, for the mean as a point of balance in the data - a sense of the importance of distance, as opposed to values - all this calculation would be inefficient and unnecessary.

6. Discussion

Several findings from this research have important pedagogical implications. First, even younger children construct average to mean a "reasonable indicator of centre". This is a key building block in learning about mathematical representativeness. Our results, unlike those of earlier studies, show that children begin developing a sense of representativeness by examining data and thinking about the relationship between data and "typicality". The idea of representativeness, then, is not something to be superimposed upon children's calculation skills: we disagree with researchers (Strauss and Bichler, 1989; Leon and Zawojewski, 1991) who claim that representativeness is a concept that develops after children have mastered the statistical aspects of average. Rather, "representativeness" in a real-life sense is an idea that emerges quite early. Building on children's intuitions, and making connections between real-life representativeness and mathematical representativeness should be a major goal of teaching statistics in the upper elementary grades.

Children's strategies for solving averaging problems (particularly when they are encouraged to use non-algorithmic ways) are varied and ingenious. Like emergent readers, young students who encounter averaging problems use a variety of contextual clues for arriving at meaningful interpretations. The shape of the distribution, including its range, modes, and middles, provides students with clues about what is representative. The data themselves, as filtered through students' everyday experience with similar phenomena, provide additional important contextual clues. Use of these contextual clues is not the same as simple egocentrism. In particular, the students in the "reasonable" and "midpoint" categories base their solutions on what makes sense in the real world as well as what makes sense in terms of characteristics of data distributions. These students are well on their way to employing average and other summary statistics in a useful and mathematically sound manner.

The results show that the introduction of the algorithm as a procedure disconnected from students' informal understanding of mode, middle, and representativeness causes a short-circuit in the reasoning of many children. A major piece of this short-circuiting is that children give up their sound intuitions about representativeness in order to embrace an algorithm that is easy to apply but makes little intuitive sense. An important question raised by our research is: under what circumstances does gaining the algorithm mean losing a meaningful concept of representativeness? It is clear that the trade-off is not one that is educationally sound. Rather than sacrifice understanding, we need to work with children's intuitive notions and help them develop ways of mapping new, varied, and richer concepts onto the ones that they already have.

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