Use of the Chance Concept in Everyday Teaching

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1. Introduction

In analyses of everyday mathematics instruction, two opposed interpretations of mathematical knowledge are becoming increasingly important. Some consider mathematical knowledge to be *objective*, while others stress the *subjective-social* conditions of classroom interaction in which mathematical knowledge is imparted. This paper presents a research study on *probability teaching* which belongs to research on *interaction in the mathematics classroom.* Whereas much research done in this field takes a *constructivist* perspective or is based on theories of *communication*, we shall focus primarily on *epistemological constraints* of mathematical knowledge in student-teacher interactions. Our specific interest will be to better understand how processes of concept *development* occur in everyday teaching and how *meaning* of mathematical concepts is embedded in social interaction.

Our underlying epistemological perspective on the nature of mathematical knowledge is that mathematical concepts possess a *theoretical* character, by which we wish to imply that the meaning of concepts cannot simply be deduced from elementary concepts already known. The concept of probability for instance, whether presented as relative frequency or as relative proportion, cannot simply be reduced to the concept of fraction. The meaning of concepts is "open" and has to be developed in the process of interpreting and extending knowledge. One important conceptual idea to help describe this feature is the metaphor of *self-reference*: mathematical concepts refer to themselves; comprehending a concept requires exploring the concept itself by extending and applying it; the complete meaning cannot be provided directly by logical deductions from former concepts.

This fundamental epistemological aspect of mathematical knowledge is in conflict with the actual methodological requirements of teaching. Everyday teaching and learning processes are organised linearly in a step-by-step manner (Bauersfeld, 1978; Steinbring, 1989). When teaching elementary concepts of probability, one tends to introduce the probability concept as an unchallengeable, clear, and conclusive definition from the very beginning. And the means of representation and activities used in the classroom are reduced to mere methodological aids to conveying the meaning of stochastic knowledge. This dilemma between the linear teaching course and the self-referent character of the mathematical knowledge being taught becomes obvious in the following micro-analysis of a short episode on teaching the chance concept.

The intention of this analysis is to reconstruct the meaning of the mathematical knowledge embedded in the classroom interaction, and to understand its relationship to the social conditions and to the conventions of teaching and learning in the classroom. While basic patterns of social interaction are mainly analysed from the communication theory perspective (cf Bauersfeld, 1985; Voigt, 1984), our discussion is intended to emphasise the *epistemological structure of the knowledge* embedded in the social interaction. Although transcribed teaching episodes can only reflect special aspects of teaching events, they can provide excellent means for decoding conventions of social interaction between teacher and students.

2. Introducing the concept of chance : a classroom episode

The following teaching episode is based on Voigt (1983) (translation by the author). In their preceding lesson, students in a fifth grade class played this game:

> In an urn there are 1 yellow, 2 green and 5 red dolls. *Rules of the game:* One doll will be drawn from the urn and then replaced. For every draw a stake of 10 Pfennig has to be paid. Following gains are possible: When drawing a green doll, 20 Pfennig will be paid and 30 Pfennig when drawing the yellow doll.

The homework for the students was to play this game 20 times and to note the results. At the beginning of the lesson, some results are written down on the blackboard (see below). When discussing this game the student Pascal offers an explaining "theory". He says: "... eh, what usually happens is that one draws 5 times the red, 2 ehm 2 times the greens and one time a yellow. ... if one would have a yellow, that is 30 Pfennig, two greens, that is 40 Pfennig, and together then 70 Pfennig, and now we have still the stake of 80 Pfennig, then this means that the player normally cannot win." (Voigt, 1983, 240 (229-237)). The teacher asks Pascal to write down his ideas on the blackboard. The main conceptual idea in Pascal's proposal is the "*ideal* mass experiment": Instead of repeating the real experiment many times, Pascal proposes to look at what will happen on the average, without explicitly explaining it in this way. Upon writing his theoretical explanation on the blackboard, Pascal says: "... and if one compares it now, so 80 Pfennig with 70 Pfennig, that's 70 Pfennig normally, hence ... theoretically wins, then one has lost 10 Pfennig." (Voigt, 1983, 240 (254-257)).

From the background of the experience with this game in the preceding lesson, with their homework and with the small piece of "theory" delivered by Pascal, the teacher and the class now start discussing the outcomes of the game. The blackboard at this stage appears as follows.

Urn: 5 red, 2 green and 1 yellow doll
Game: Drawing a doll with replacement of the doll
Stake: 10 Pf per draw /

Gain: 20 Pf when drawing a green doll
30 Pf when drawing the yellow doll

Stake	20	20	20	20	20	20	20	20	20	20	20	20	200
Gain	21	24	26	27	19	20	13	23	25	17	~~29~~	~~22~~	215

2 green	40 Pf		dolls
1 yellow	30 Pf / 70 Pf	8 Games	$\hat{=}$ 80 Pf stakes

Blackboard Image 3

T: What's the others' comment to that? ..... Marc (*T stands on right side of blackboard, walks out of the picture.*)
Mc: I think if you have such an experiment and then somebody comes with such a theory and you have tested that before you will not say this theory is right as you tested it out before and quite the opposite has come out. Then you don't believe that.
T: Yes. And what is it what you would rather believe. (*T stands on the right side of the blackboard.*)
Mc: Well, that of the experiment where the result was stake two hundred and gain two hundred and fifteen.
T: Hm. That is you rather believe that here then the idea of eh. You rather believe that than the idea Pascal had. (*T points to the data table, then to blackboard image 3.*)
Mc: Yes. I know now that Pascal's idea is right only only if ... if I were somebody who just had made the experiment and did not know anything about the idea, then Pascal comes with his idea and says, so this here is right, then nobody will believe him then he must justify this somehow and show it.
T: Mhm. That is your attitude is Pascal will have to join the game and lose all the while ... as a player. (*T advances between desks to Marc, smiles.*)
Mc: Well, Pascal would have to demonstrate this somehow that this is really right.
T: In the game itself?
Mc: Yes.
T: Mhm ... Markus. (*T walks backward to his desk.*)

Ma: Pascal's theory is right but only the ... difference is too small, well only ten Pfennig eh that it is well possible that the player will win in practice.
T: Hmhm ... why is it that the player can well win in practice ... as happened here ... Lars. (*T raises his finger to his mouth.*)
La: Because you don't exactly with eight draws, if you draw eight times that you will always draw five red ones and two green ons and one yellow one. You can also sometimes draw green all eight times, and then the player has won.
NSS: (*murmuring*)

S: Yes that is really all chance, if you now draw always the red ones then the player has lost, but if you draw the yellow ones then has always the yellow ones then the player has won ... it quite depends on what you draw.
T: Andreas.
Ad: Eh you always put back the dolls and this is why it is rather improbable that somebody draws with all eight draws precisely the ... every time a doll which, well precisely the dolls which are there and two times the yellow one.

T: Speak a bit louder, Kathrin.
Ka: If you play ten games of this kind (...) tries them out (...).
T: Mhm. Lutz.
Lu: Yes because because if you draw a yellow one you won't do this often, but you win thirty Pfennig (...).
T: Mhm. (*T moves to right side of blackboard, finger to his mouth.*)
Lu: Otherwise this would hardly be so that (...) player wins.
T: How would you circumscribe that now, or what would you say if I drew the a yellow ... the yellow doll seven times in succession. (*T walks to and fro between the benches.*)
S: Chance.
S: Luck.
S: This is chance.
T: How do you call such a thing?
S: Cheating.
T: Hermine.
He: Chance.
T: Yes, this is chance. That is the very thing Mirco said. By chance, this result comes out. There is chance in that ...

3. Analysis of the teaching episode

The teaching episode can be subdivided into the following six phases.

(i) *Phase 1 (263-265): Presentation of the outcomes of the game at the blackboard* (see Blackboard Image 3).

(ii) *Phase 2 (266-276): Marc's explanation.* Marc doubts that Pascal's theory is appropriate or correct, because the trials show totally other results. The teacher reinforces this doubt saying that Marc seems to prefer believing the empirical outcomes rather than Pascal's theory.

(iii) *Phase 3 (277-288): How Pascal can justify his theory?* Here again it is doubted that Pascal's theory is appropriate. Formally it is correct, Marc now concedes, but it is not possible to believe him. First Pascal must justify why it is valid. Marc says: "... well, Pascal would have to demonstrate this somehow that this is really right." (285/6)

(iv) *Phase 4 (289-292): Markus's explanation.* Markus remarks that Pascal's theory is correct. The difference is only small, he says, and hence one could conclude that here is no contradiction.

(v) *Phase 5 (293-309): The restriction of Pascal's theory.* Lars expresses the opinion that with eight draws one can obtain any combination of yellow, green, and red dolls, and the combination proposed by Pascal is only one possible. Hereupon, one student offers "chance" as an explanation and argues implicitly that Pascal's "event" is as random as the event "eight times red" or the event "eight times yellow". The student Andreas also reinforces this explanation by "chance" and tries to reject Pascal's theory because his proposed combination is very improbable.

(vi) *Phase 6 (317-336): Stabilisation of "chance" as the accepted pattern of justification.* The teacher takes up the argument that even events with a very small probability might occur in order to reach his planned solution: the discrepancy between experimental results and theoretical expectations must be resolved by "chance". The extreme event "the yellow doll seven times in succession" is taken as touch-stone to guarantee this pattern of explanation. One student's audacious remark "cheating" (332) is not considered at all.

The presentation of the experimental results together with Pascal's theory on the blackboard lead to a relatively broad and open discussion. It moves between possible theoretical arguments and empirical phenomena. In this frame, the following questions come up:

(i) Is the theory proposed by Pascal justified?
(ii) Is this theory correct?
(iii) How could it be interpreted with regard to the situation at hand?
(iv) How shall the difference found here between theory and practice be judged?

The original openness of the classroom interaction is increasingly restricted in the course of teaching. A student makes the first restriction of Pascal's model by confronting the event "getting exactly 5 times red, 2 times green and 1 time yellow with eight drawings" with the event "getting every time green with eight drawings" (293-309). The teacher and other students take up this idea and introduce "chance" as the only valid pattern of justification. The teacher strengthens this pattern stating the hypothesis "... what would you say if I drew ... the yellow doll seven times in succession" (325-327). Faced with this extreme example students shout "Chance! Luck!" etc. The teacher uses this *contraction of the frame of interpretation* to codify "chance" as the only legitimate pattern of justification for the difference between theoretical predictions and empirical results (cf Maier and Voigt, 1989). The routine mechanisms of the funnel pattern (cf Bauersfeld, 1978) clearly become apparent in this classroom interaction.

Later in this lesson, this justification freezes in the concept of "pure chance". At the end, "chance" no longer expresses a positive conceptual idea suitable for exploring, understanding, and solving a stochastic problem. It has degenerated into a substitute for justification, which serves to deny the importance of the difference between theory and empirical facts in probability.

Pascal's model is more and more deprived of its theoretical character. The perspective of modelling in a simple "ideal" set-up what will be expected in a real mass experiment gradually changes. By contrasting Pascal's idea with other possible events, as "eight times green" (Lars, 297), "eight times red" (S, 301) or "eight times yellow (S, 303), the model becomes a real event like others. And when even the teacher makes use of such a reduction of the model to an actual outcome, then this confirms that Pascal's theory itself must be improbable and that the chance concept alone is able to explain all these mysterious differences. But surprisingly, later, the teacher takes up Pascal's model, now with a different perspective: to calculate the average loss or gain when drawing eight times. In one way or another, acceptance of the argument seemingly depends on the methodological purpose pursued.

4. Patterns of developing social meaning in the teaching process

The observations of the preceding lesson clearly show the intentions with which the experiments were performed. They also contain indications about the ideas hidden behind the chance concept. At the beginning, "chance" is implicitly conceived of as a *reverse-concept*: chance is present in situations where no physical law can be observed.

The idea that chance covers the non-regular, the irregular and lawless aspects of reality already refers to the fundamental relationship between theory and experiment: deterministic, physical laws and definitive predictions are elaborated within the theory and in theoretical models; real phenomena and experimental outcomes can be observed in practical situations.

An important consequence of this preliminary definition is that real chance events do not occur with absolute certainty according to deterministic laws, but only with a certain probability. It opens relations to other everyday ideas of chance, as for instance "having good luck!" or "having bad luck!": contrary to the expectation predicted by theory, a favourable or a harmful event with a very small probability nevertheless has occurred.

Aspects of such an intuitively-based idea of chance can be observed in the course of the preceding lesson. The classroom interaction with playing the game and evaluating the experimental outcomes constitutes a social medium for further conceptual developments. It still contains unconnected elements on the following levels:

(i) on the *object-level:* the experimental drawings, the determination of relative frequencies, etc.;
(ii) on the *symbol-level:* calculating probabilities (gain and loss) with the help of an elementary stochastic model.

This social context opens up two major possibilities for the further conceptual development of chance:

(i) *a reduction of the chance concept to a formalised, conventional label;*
(ii) *an expansion of the chance concept to a means of analysis for the relation between experimental situation and stochastic model.*

In this lesson - and this is the starting point - a contradiction between theoretical prediction and empirical observation is stated: Pascal's elementary theory predicts a loss in the game, whereas the actual playing seemingly results in a gain.

How is this contradiction handled? In the end, a methodological reduction is presented as a solution: the reverse-definition of chance as non-existing regularity has become more and more accepted. A theoretically impossible gain has occurred, but because this empirical phenomenon is not subject to any causal law, the standard justification is valid in this case as well: it is "chance" in the form of "having good luck" which serves as the conventional explanation for this observed contradiction.

The teacher makes this pattern of justification an absolute one by using the hypothetical event (325-327) "drawing the yellow doll seven times in succession". This thought experiment could be an opportunity for changing the direction of argumentation and it could serve for a qualitative further development of the chance concept: if a very improbable event has occurred this could perhaps indicate that there is something wrong in the whole process of experiment and analysis. This could lead to a "*theoretical inversion*" in the analysis of the complex process consisting of implicit assumptions, arrangements for the experimental set-up, observing and measuring outcomes, and comparing them with theoretical predictions calculated according to stochastic rules and models. In principle, all elements of the process have to be questioned when a very rare event is observed. The inversion of the question of justification would transform the epistemological status of the chance concept from an empirical label to a theoretical concept. Then the focus could be directed at the relation between the *practical conditions* for performing the experiment (dependent drawings, independent drawings, drawing with replacement, drawing without replacement, etc.) and the corresponding *theoretical conditions* for constructing the stochastic model (combinatorical combinations of favourable events, multiplication of probabilities, idealised character of the model, etc.).

The *methodological universalisation* of chance as the counterpoise to causal laws leads to a disregard of the difference between the experimental situation and the stochastic model. However large the differences observed between theoretical predictions and empirical outcomes may be, the methodological universalisation of chance will always be able to explain this contradiction as something quite natural.

Instead of reducing the chance concept to a methodologically universal definition, many of the students' contributions offer opportunities to unfold its development. The social interaction constitutes an open and rich environment for potential generalisations of the chance concept. Within the teaching/learning process with its communicative patterns and routines (Maier and Voigt, 1989) *methodological universalisation* compels a conventional reduction of the intuitive concept: chance is conceived of only as the opposite of regularity. But the increasingly clear contradiction between Pascal's theory and the empirical outcomes could in principle have led to another development: of unfolding chance, with the help of stochastic independence, as a means of controlling the balance between stochastic model and random experiment. According to the Bernoullian model, the outcome of the game played by the students has a very small probability, of less than 0.1%; thus it seems plausible to assume some discrepancy between the way the experiment was performed and the assumed conditions underlying the stochastic model.

Such *re-analysis* of experimental conditions and theoretical assumptions plays a fundamental role for the development of basic stochastic concepts. Advanced stochastic techniques can be used in a way of *self-application* or of *feedback* to re-analyse the experimental situation of the actual classroom teaching.

In this teaching episode the social constraints of the classroom discourse lead to a very specific form of mathematical knowledge for school. In particular, the routine funnel-shaped classroom interaction between students and teacher produced a deformed mathematical concept of chance on the micro-level. Intuitive and naive representations of mathematical pre-concepts are standardised in form of universal definitions. Such universalised concepts appear clear and unambiguous from the very beginning. Their epistemological structure does not contain any circularity or theoretical self-reference; they have become *fixed generalisations* cemented by methodological convention.

The sequential course of teaching processes is in conflict with the self-referent structure of mathematical knowledge. If intentions of direct and immediate instruction dominate the interactive classroom process then the deductive structure of the teaching processes deforms the epistemological structure of mathematical knowledge. "The mathematical logic of an ideal teaching-learning process ... becomes replaced by the social logic of this type of instruction." (Bauersfeld, 1988, p.38). The methodological transformation of mathematical concepts removes their theoretical nature.

How can the epistemological structure of mathematical knowledge and the structure of teaching-learning processes be made compatible in a way that knowledge is not simply reduced to mere methodological convention? The solution cannot be to directly adopt the scientific epistemology, but to constitute a proper knowledge-epistemology in the social interaction. The metaphor of "self-reference" should be used seriously, not only for knowledge, but for interactive classroom processes as well. Common social understanding and development of knowledge requires an explicit, interactive feedback-structure for checking, improving, and modifying the comprehension of mathematical concepts. The experiments with the game the students played should not be used simply for motivation or as the start of a step-by-step progress towards the intended goal. This concrete context of their experiences with the game is a fundamental source for the students, which has to be maintained throughout the whole process of developing the chance concept. Here, the self-referent social process can start in a manner analogous to that required by the epistemological structure of the stochastic concepts.

The deformed mathematical knowledge with its methodologically universalised concepts causes a reduced *subjectivity* of knowledge. Reduced knowledge becomes more or less arbitrary for the learner, because it seems to be a pure convention or completely determined by the teacher's methodological intentions. Only "theoretical knowledge" - even on a very elementary level - is open for individual and personally subjective approaches. Especially stochastic knowledge requires direct subjective decisions and interpretations. Inversion of the theoretical analysis as the beginning of conceptual generalisation is closely linked with the involvement of the student. The learner has to decide how to take the statement: "There is something wrong in the relation between theoretical model and empirical observation!". It is the self-referent character which makes knowledge alive and allows the student to participate in this developmental process. Such an understanding of *theoretical knowledge* will permit the re-establishment of an appropriate balance between objective and subjective aspects of knowledge in processes of teaching, learning, and understanding.

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