Application of ordinary least square method in nonlinear models

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1. Introduction

If among economical phenomena exist nonlinear interconnections, they are expressed with help of corresponding nonlinear functions. Teaching the regression analysis for the economic science students one of the important topic is method of Ordinary Least Square (OLS) and their application in the nonlinear regression analysis. Based on experience at Latvia University of Agriculture, the illustrated approach of teaching of nonlinear regression topics for undergraduate students in economics is presented (Arhipova I., 2006). Teaching statistics on regression analysis, students usually have problems with interpretation of the transformed regression model parameters significance (Arhipova I., Balina S., 2004). The tasks of teaching using OLS in the nonlinear regression analysis are discussed.

Models which are nonlinear in parameters, in sense, that by suitable (log) transformation the models can be made linear in parameters. In this case method of Ordinary Least Square (OLS) has been used for transformed equations. However, it is possible that the stochastic term has different variability related to dependent variable: additive or more random and unpredictable. Depending on the stochastic term variability the nonlinear models can be intrinsically linear (in-parameter) regression models (in the sense that by suitable (log) transformation the models can be made linear in the parameters) and can be intrinsically nonlinear in parameters.

If for the linear models and models nonlinear in variables the least-squares criterion of minimization has been applied to initial (original) variables, then for the models nonlinear in parameters the least-squares criterion of minimization it has to be applied to transformed variables, for example *LnY*. However, the parameters estimation for the transformed models OLS is biased. It means that, although nonlinear models can be transformed into linear regression models and can be estimated by OLS, we have to be careful about the properties of the stochastic residual term that enters these models. As the preceding analysis shows, it is necessary to pay attention to the residual term in transforming a model for regression analysis. Otherwise, a formal application of OLS to the transformed model will not produce a model with correct statistical properties. The examples of the different nonlinear models and the application of OLS are considered, as well the transformed models estimated parameters has been compared.

2. The least-squares criterion of minimization for linear and nonlinear models

Consider the following regression model (1).

$$Y_i = \beta_1 \cdot \beta_2^{X_i} \cdot \varepsilon_i \qquad (1)$$

Models like (1) are intrinsically linear (in-parameter) regression models in the sense that by suitable (log) transformation the models can be made linear in the parameters β_1 and β_2 (Note: Model is nonlinear in β_1 and β_2). In this case we assume that that the error variability isn't constant at all X_i, that is, that the error isn't additive. Most likely, there is more random and unpredictable fluctuation at the different levels of X_i. We assume that (2)

$$\prod \varepsilon_i = 1 \quad or \quad \sum Ln\varepsilon_i = 0 \qquad (2)$$

Taking the logarithms on the both sides of the equation (1), we obtain equation (3).

$$LnY_i = Ln\beta_1 + X_i Ln\beta_2 + Ln\varepsilon_i \qquad (3)$$

Using OLS for the sample regression function under the minimum criterion (4)

$$\sum (LnY_i - Lnb_1 - X_i Lnb_2)^2 \qquad (4)$$

the process of differentiation yields the following normal equations (5) for estimating β_1 and β_2:

$$\begin{cases} \sum LnY_i = n \cdot Lnb_1 + Lnb_2 \cdot \sum X_i \\ \sum X_i \cdot LnY_i = Lnb_1 \cdot \sum X_i + Lnb_2 \cdot \sum X_i^2 \end{cases} \qquad (5)$$

Solving the normal equations simultaneously, we obtain (6)

$$Lnb_1 = \frac{\sum LnY_i}{n} - Lnb_2 \cdot \frac{\sum X_i}{n} = \frac{\sum LnY_i}{n} - Lnb_2 \cdot \overline{X} \qquad (6)$$

Let $\overline{X} = 0$, than (7)

$$Lnb_1 = \frac{1}{n} \cdot \sum LnY_i \quad \text{or} \quad b_1 = \sqrt[n]{Y_1 \cdot Y_2 \cdot ... Y_n} \qquad (7)$$

It means that parameter β_1 is the geometric mean of variable Y. To use the classical normal linear regression model, we have to assume that $Ln\varepsilon_i \sim N(0, \sigma^2)$. Therefore, when we run the regression (3), we will have to apply the normality tests to the residuals obtained from this regression (Gujarati Damodar, N., 1995).

Now consider the following regression model (8). This model is intrinsically nonlinear in parameter. In this model we assume that the error variability is independent of X_i, that is, that the amount of residual error variability is the same at any X_i.

$$Y_i = \beta_1 \cdot \beta_2^{X_i} + \varepsilon_i \qquad (8)$$

Using OLS for the sample regression function under the minimum criterion (9)

$$\sum (Y_i - b_1 \cdot b_2^{X_i})^2 \qquad (9)$$

the process of differentiation yields the following normal equations (10) for estimating β_1 and β_2:

$$\begin{cases} \sum 2(Y_i - b_1 \cdot b_2^{X_i}) \cdot b_2^{X_i} = 0 \\ \sum 2(Y_i - b_1 \cdot b_2^{X_i}) \cdot b_1 \cdot X_i \cdot b_2^{X_i} / b_2 = 0 \end{cases} \qquad (10)$$

It isn't possible to solve these normal equations analytically, and in this nonlinear case, nonlinear ordinary least-squares estimation can be performed iteratively using a linearization of the model with respect to the parameters. It is clear, that the parameters β_1 and β_2 estimation have got from the minimum criterion (8) are biased from the estimation of the parameters β_1 and β_2 have got from the minimum criterion (4).

We have to be careful about the properties of the stochastic error term that enters this model. For hypothesis testing, we have to assume that for the model (8) stochastic residual term ε_i follows the normal distribution. But for the model (1) and its statistical counterpart (3) we have to assume that $Ln\varepsilon_i \sim N(0, \sigma^2)$ and respectively stochastic error term ε_i must follows the log-normal distribution with mean (11) and variance (12):

$$e^{\sigma^2/2} \qquad (11)$$

$$e^{\sigma^2}(e^{\sigma^2} - 1) \qquad (12)$$

3. Graphical application of OLS to the transformed model

Let examine the trend in Latvia income and identify the impact of income changes on private consumption distribution. Analysis is based on gross private consumption and average income per capita Latvia Central Statistical Bureau data for the time period from 1996 to 2005 in real 2000 year prices for the different regions of Latvia (Statistical Yearbook of Latvia, 2006). The following hypothesis is chosen for regression analysis: gross private consumption distribution depends on average income per capita. In this task special attention is paid to the interpretation of linear and transformed models parameters.

The relationship between consumption and income is one of the most important issues in macroeconomic model building and forecasting (Takala K., 2001). The following model was chosen for the consumption modeling depends on income (13), where Y_i is Latvia gross private consumption in mln LVL and X_i is the average income per capita in LVL:

$$Y_i = \beta_1 \cdot X_i^{\beta_2} \cdot \varepsilon_i \qquad (13)$$

Taking the logarithms on the both sides of these equations, we obtain equation (14).

$$LnY_i = Ln\beta_1 + \beta_2 LnX_i + Ln\varepsilon_i \qquad (14)$$

Using OLS for the sample regression function under the minimum criterion (15)

$$Z_1 = \sum (LnY_i - Lnb_1 - b_2 LnX_i)^2 \qquad (15)$$

the following parameters β_1 and β_2 estimations were obtained (16):

$$b_1 = e^{4,38} = 79,95 \qquad b_2 = 0,78 \qquad (16)$$

The regression results show that parameters $Ln\beta_1$ and β_2 are both significant (p-value<0,05) and gross private consumption is depended on average income per capita. The slope parameter β_2 measures the elasticity of Y with respect to X, that is, the percentage change in Y for a given percentage change in X. From these results we see that implying that for a 1 percent increase in the real average income per capita, the gross private consumption increases by about 0,78 percent (figure 1).

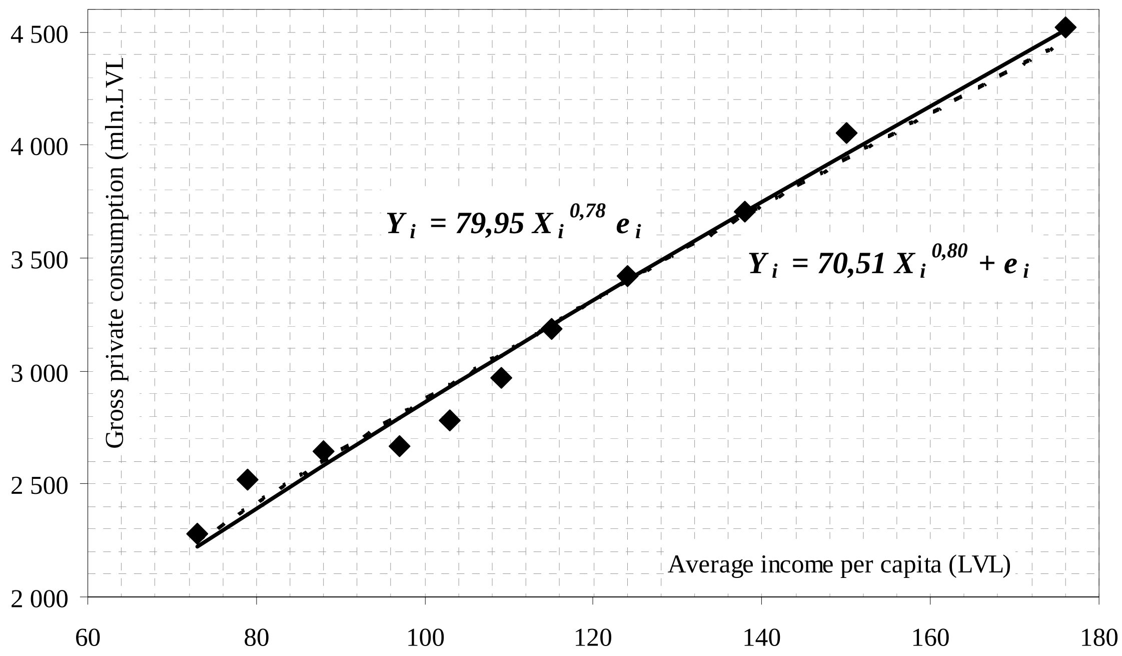

Figure 1. Consumption and income relationship using models with additive and unpredictable residual term

Let assume, that the following model was chosen for the consumption modeling (17):

$$Y_i = \beta_1 \cdot X_i^{\beta_2} + \varepsilon_i \qquad (17)$$

Using OLS for the sample regression function under the minimum criterion (18)

$$Z_2 = \sum (Y_i - b_1 \cdot X_i^{b_2})^2 \qquad (18)$$

the following parameters β_1 and β_2 estimations (figure 1) were obtained by the iteration process (19):

$$b_1 = 70,51 \qquad b_2 = 0,80 \qquad (19)$$

The conclusion is that both minimum criterions $Z_1(b_1;b_2)$ (15) and $Z_2(b_1;b_2)$ (18) have biased minimum solutions (figure 2). The problem is that when we run the regression (13), we will have to apply the OLS for the intrinsically linear (in-parameter) regression model, but when we run the regression (17) than we will have to apply the OLS for the model intrinsically nonlinear in parameters.

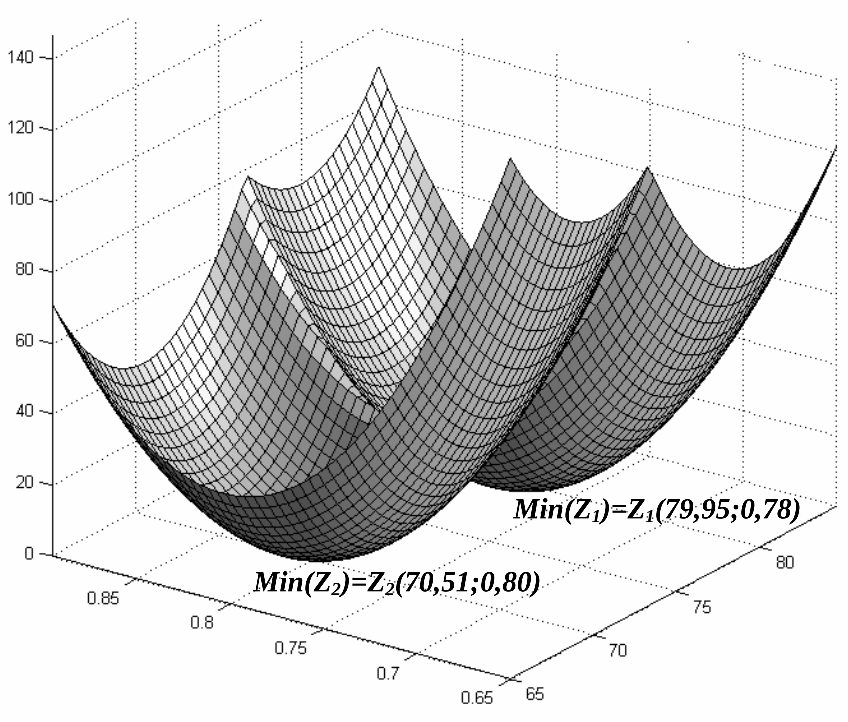

Figure 2. The comparison of the models minimum criterions solutions

4. Conclusions

The application of OLS allows the students to obtain clear interpretation of the statistical models as well as to help them better understand the transformed models parameters interpretation. Students get clear explanation about the OLS field of usage in the tasks of nonlinear regression, as well the concepts about the statistical properties of transformed models. The preceding analysis demonstrates for students the properties of the stochastic residual term that enters these transformed models.

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