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THE CLASSIC DEFINITION OF PROBABILITY AND FRACTIONS: DIFFICULTIES AND POSSIBILITIES AT THE PRIMARY SCHOOL LEVEL

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1. Introduction

The concept of fraction represents, a moment of difficulty both for teachers, from whom it requires adequate teaching abilities, and for children, who must be highly motivated to understand it. It is not rash to state that even in the scholastic stages following the primary school level the approach to fractions requires special attention on the part of teachers and students. Fractional notation expresses different "mathematical moments" which obviously go beyond the traditional teaching concept of the *whole* to the *part*. The educative process at the primary school level traditionally starts from this approach and with this content, with the idea of latest broadening its fields of application. In some of these the same notation is subsequently used. In Italian primary school syllabuses for 1985, the goals outlined for the first cycle (first and second years) call explicitly for the use of the fraction symbol in the classic terms of the concept of *whole/part*:

"... with the aid of an adequate number of objects, calculate the reciprocal connection double/half, triple/third, quadruple/fourth, and so on..."

Among the goals of the second cycle (3*rd*, 4*th* and 5*th* years) pupils must :

"... find the fractions representing parts of suitable geometrical figures, sets of objects or numbers; conversely, given a fraction, find the corresponding part in appropriate geometric figures, sets of objects or numbers, paying special attention to decimal places. Compare and put in order the simplest fractions, using properly the number-line ...".

In the opinion of those who drew up the syllabus, this was the approach most suitable to the cognitive and psychological development of the child, who has already concluded his or her pre-operational period. It should form a solid foundation on which to build subsequent meanings of fraction symbolism. The extension of curricula to include probability, similitude and so on, does not appear to take for granted the priority given to the *whole/part* approach. In fact opens it to discussion and should stimulate research and experimentation to verify the validity of this assessment of priority. In school usage, the fraction symbol, in its meaning of whole/part, is found to be the most commonly accepted, and seems to establish a fairly natural relationship between child and concrete reality. To divide and recompose objects responds to the needs of children to act with and on things, and consolidates reversible thought. The mental capacity to arrive at a solution following determined pathways and then return, by means of analysis, to the starting point, represents the fundamental characteristic of reversible and operational thought. The development of an ever more complex society and the necessity of representing it symbolically obliges us to educate future generations in the decoding of different situations, even when they are expressed with the same symbols.

2. Fractions are difficult

That fractions are difficult is indisputable, and has been affirmed by others in the past (Hart, 1981). Historically, the traditional approach to teaching fractions has required the setting up of an itinerary calling for a more or less extended stage of manipulation and execution. Furthermore, psychological research into the learning process has shown that while it is true that perceptive experience helps in understanding this concept, it may also generate a certain functional fixity and form an obstacle to the enlarging of successive knowledge. The measurement of uncertainty is included, and quite rightly so, in a cognitive perspective that must be imparted to children at primary school level. How can we approach, gradually but systematically, the teaching of the analysis of probabilities? As is well known, this kind of analysis is expressed with fractional symbols signifying different things in different contexts. It is to be emphasized that the passage from action/perception to the fractional representation of numerical quantities has always represented a moment of crisis both for learner and teacher. Our experience has shown that after suitable drilling, children are capable of responding correctly to requests of the kind:

a) "Color two thirds of the figure"

b) "Draw a line around one third of these circles"
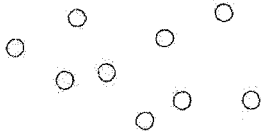
It goes without saying that the first exercise is much easier than the second, which implicitly requires the children to count the circles, divide them and single out the part to draw a line around. We should point out that this example also contains a perceptive difficulty, since the numerator of the fraction 1/3 does not correspond to the number of balls, which in this case should be three. This is quite different from the first example, in which children were asked to color two clearly visible parts. Still remaining within the limits of the concept of a "pie" (a rectangle or any other object that can be divided into equal parts) the fractional symbol maintains its undeniable difficulty.

3. Fractions and the classic definition of probability

At this point we ask: is it useful and effective to start from the whole/part approach in tracing didactic itineraries leading to the differentiated interpretation of the fraction symbol? In other words, is this the only possible way in which to reach an understanding of different realities? From the pedagogical viewpoint, it is not really right to use the fraction symbol if diversified didactic itineraries, one for each different situation, are not designed previously so as to ensure that the child comes up with a correct interpretation of reality. Experimentation with new teaching strategies can no longer be postponed if we wish to avoid gaps and cognitive lapses. The "new" use of the fraction symbol, with the inclusion of probability in the 1985 syllabuses, not only adds new difficulties to the traditional concept of fraction, but also represents a further obstacle between the learner and the concept of probability (Fischbein, 1969). When a child is asked what the probability is of an even number coming up on throwing a die, it is not a foregone conclusion that

his or her answer will be 1/2. In fact, the representation in terms of a fraction of a probabilistic situation from the teaching viewpoint is not completely referable to the fractional representation of whole to part. At the end of their primary school careers, children should be able to understand and apply the classic definition of probability after graduated exercises (Boffa and Caredda, 1990). This means that they must be able to arrive at qualitative and quantitative evaluations of probability in simple, and obviously symmetrical, aleatory situations.

According to the classic conception, in order to express quantitative evaluations of probability one must be capable not only of performing combinatorial analysis, but must also know how to use the fraction symbol adequately. In this context, the fraction symbol is not an operator and does not express the division of the whole into parts, but assumes the function of measurer of uncertainty. If it is true that the whole/part concept is accessible to children, it is also true that it involves not only a certain difficulty in mastering it, but (Varga, 1969a,b) also a mental stereotype that may hinder future uses of this symbol.

Situations of uncertainty, for the very reason that they are uncertain, generate phenomena of different kinds. Firstly, we are faced with problems of a psychological nature. When we are required to make choices or come to a decision in situations in which we do not fully understand all their aspects and implications, our emotional states lie at the base of mostly "unreasoned" choices. Children who have not yet perfected their processes of maturation require both educative support that helps them to control their states of anxiety in situations of uncertainty as well as didactic support that accustoms them to making their evaluations as objectively as possible when exchanging and interchanging information. In the period in which children attend primary school they must learn to recognize the fact that in interpersonal relations the viewpoints of their peers have the same value as their own. It is in fact in this continuous confrontation, as well as in an adequate education, that it is possible to arrive at acceptance of the opinions of others and at the realization that others are of the same worth. Thus the capacity to concede merit to other viewpoints is the result of a long process that must not, however, be too long if we wish to arrive at a harmonious development of the personality. How closely affective-emotive development is connected with cognitive development is now an established fact in the literature of psychology, which stresses that acquiring a positive image of oneself is indispensable in overcoming failed attempts and negative results at an acceptable level of frustration. In school situations dealing with the themes of probability, it becomes necessary for children to be capable of controlling on the one hand the emotive state activated in aleatory situations and on the other

hand of being able to express objective evaluations uninfluenced by personal desires and preferences. Concerning probability, the goals of the first and second years are:

"In problematic situations taken from real life and play, use coherently and in the most meaningful way the expressions: 'perhaps', 'it's possible', 'it's certain', 'I don't know', 'that's impossible', and so on".

To trace a didactic itinerary that goes beyond the acquisition and use of the concepts of "true" and "false" to the concept of "possible" is certainly not an easy task. Teaching, to be effective, must bring ludic activities into play since they are congenial to the age of the child and its way of expressing itself and because their pedagogic value is universally recognized. This value is recognized in all scholastic disciplines and especially in mathematics, within which it not only facilitates the understanding of concepts, but also succeeds in verifying the theoretical assumptions referring to learning processes.

In aleatory situations, ludic activity maintains its pedagogic value, but it appears to us that it assumes a different role (Caredda and Puxeddu, 1990). While in common mathematical situations play verifies and completes theoretical assertions, in aleatory situations verification is lacking, but it gives rise to a cognitive conflict representing, in this context, the salient element in the learning process. The originality of the educative function of play lies in the "overcoming" of certainties and in the assumption of situations of uncertainty in cognitive structures. In ludic activities referring to the classic conception of probability (such as the throwing of a die) the outcome does not confirm expectations, thus often generating in the child the idea that the coincidence, or the lack of coincidence, between expectation and outcome is the result of good or bad "luck". The usefulness of the teacher's intervention consists of teaching that the rolling of "two" with the die is not the result of Johnny's prediction or of his "bad luck" if the two doesn't come up, but one of six equally possible outcomes.

Having attributed to play the role of stimulator of cognitive conflicts does not mean that we have eliminated other difficulties that are also present in situations of uncertainty. The pre-eminent function of language in teaching is universally recognized through its being present in all cultural contexts, to such an extent that it is rightly considered a "facilitator" in learning and teaching interactions. For it to be such, not only is semantic sharing a necessity, but also the two-way grasp of consciousness of what one intends to say. Acquisition in aleatory situations in terms such as: "perhaps", "it's possible", "it's certain", and so on must be placed within a learning process that must be indivisible linked to the

use of terms that are aleatory in daily experiences. The learning of terms that "speak of probability" must go beyond the scholastic and be inserted in a more general cognitive sphere. Teaching must move towards the overcoming of a linguistic separation between aleatory classroom situations and those of daily life and must, above all, see to it that the child accepts situations of uncertainty not only because they enrich his or her experiences (beyond the true-false dichotomy) but also to set him or her out in the direction of an evaluation of events. The use of the terms "perhaps", "it's possible", "it's certain" must be deprived of all emotive and affective connotations only to become more and more reasoned and objective in the course of a suitable didactic itinerary (Caredda and Puxeddu, 1991a).

4. Hypotheses concerning what to do: some considerations

In the present-day situation the use of the fraction symbol to express different meanings and the inclusion of the classic definition of probability in the whole/part fractional concept appear to us to be two possible approaches. Undoubtedly, the first approach requires the tracing of a gradual and continuous itinerary starting from the first year of primary school. The introduction of the generalized use of the fraction symbol in a teaching context would, in our opinion, lead to the elimination of those "certainties" that must in any case be overcome. At this point, we do not feel that a reference to Piaget (see Piaget and Inhelder, 1951) is out of place if we consider that it is in the period of concrete operations that the child succeeds in grasping and controlling at least two attributes of objects and situations. A didactic itinerary that fails to respect this rhythm of learning would lead to a cognitive development lacking, either wholly or in part, the logical tissue that is at the base of all human knowledge.

Therefore, the fractional symbol can be used effectively in "reading" different situations, as long as the child's learning processes are respected and graduated and continuous didactic itineraries are laid out. Such an idea may "frighten" primary school teachers, not only because, as we said before, fractions are difficult, but also because it will require them to widen their methodological and teaching horizons. But it should not be particularly difficult to get children to understand the efficacy and immediacy of a symbol, even when it is used to describe different situations. Today, the linguistic code in use is restricted, contrary to what Bernestein has stated (see Bernestein, 1970), as the result of sophisticated mental processes borrowed from computer science. We should not

therefore be surprised if a single symbol is used to describe several different situations. The sharing of context, without which it is extremely difficult to understand others and to understand in general, is obviously indispensable. Linguistic experience shows that the same word in different contexts acquires different meanings. A primary school child, if he or she is aware of the different contexts, understands that the word "moon" assumes different meanings in the sentences "He wants the moon", "I win once in a blue moon" and "He's been mooning about all day". Why not do the same with the fraction symbol? There is, however, the real and objective difficulty with the diversified use of the fraction symbol, especially in consideration of the peculiarity of the learning processes the child goes through at the age at which he or she attends primary school. To this we must add difficulties connected with the periods of time needed for planning and developing the project itself. In the meantime, the continuity of educative interventions must be ensured. The didactic pathway must not be delayed or interrupted, and all useless and sterile superimpositions must be avoided by supplying a graduated and pedagogically planned procedure. In our view, the child of today, who is exposed to a large number and variety of signs and symbols and who knows how to play videogames should not find it difficult to recognize different contexts and interpret them with identical symbolic expressions.

If we asked children to represent the following cyclic cadence of shapes

and told them that such a cycle could be expressed not only with the numbers 2 and 3 (two representing the squares and 3 the circles) but also with the fractional symbol of two over three (2/3), it should not cause them any great difficulties. The same thing can be said for rhythmic sequences, which are often used in the first year in teaching reading and writing. The idea of blowing a whistle twice and beating a drum five times can be expressed both as (2, 5) and as 2/5. It should not be difficult, in situations describing spatial localization such as "Johnny went to see a play put on by his fellow fifth-graders. He sat in the second row in the third seat", to teach that this can also be expressed using the fraction symbol two over three (2/3).

The use of the fraction symbol in diverse contexts must not cause us to lose sight of the close connection between the symbol and the very concept of fraction, which represents the simplest expression of the subdivision of a

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whole into different parts. In elementary school syllabuses it is explicitly stated that geometric figures can be cut out or suitably colored, and that it is necessary to be able to recognize the complementary fraction, the proper fraction and the apparent fraction. The concept of fraction together with its graphic representation represents the most commonly known expression, although, as we have argued, the fractional symbol may express other concepts.

The second itinerary calls for the inclusion of the classic definition of probability in the whole/part acceptation of the fraction symbol. Despite the apparent difficulty this entails, we believe it represents a feasible itinerary, not least because it requires neither long-term planning nor the need for the conditions indispensable for the diversified use of the fraction symbol. Thus the introduction of the measurement of possibility also means to enlarge conceptually a process that is in any case already in step with the fraction itself. It is not so necessary to plan a special and different teaching approach as it is to familiarize children with the solving of problems having more than one solution. The classic problems with the drawing of the cartesian product can be a good beginning in orienting children towards finding as many solutions as they can. Not only this, but the use of the language of sets and the use of these in scholastic practice should help children to conceptualize "that the six sides of the die are a set and that the three even-numbered sides are one of its subsets". Introducing the measurement of the possibility of a certain event occurring fits into the fraction concept of whole/part without recourse to any teaching expedient. The idea is to propose and have the children verify that the six sides of the die are like a whole pie and that the three even-numbered sides represent, for example, three slices of pie that have been eaten. The symbolic representation is perfectly identical: 3/6 and thus 1/2. The setting up of teaching programs, and not only those concerning mathematics, must take into account appropriate situations that can aid in developing the capacity to compare and diversify knowledge. In the case of mathematics, this is generally favoured by combinatorial analysis, which systematically analyzes all possibilities. In our opinion, the operational proposals presented herein reflect the graduality and continuity necessary to ensure correct and meaningful assimilation (Boffa and Caredda, 1990).

Example 1: "During the spring track and field events, Al, Bert and Charlie got onto the podium. What positions could they have taken there?"

Example 2: "In a small lottery with only ninety tickets, the prize will be awarded to a ticket bearing a number from 1 to 90. Michael bought all the tickets numbered with multiples of five. Later, Dotty bought all the tickets numbered with multiples of three that were still left. How many tickets did Michael buy?

What are Michael's chances of winning the prize? Who has a better chance of winning, Michael or Dotty?"

Example 3: "A 40-card deck of playing cards has been divided into two stacks, one containing all the face cards and the other containing all the numbered cards. Both stacks have been carefully shuffled and placed on the table face down. From which of the two stacks are you more likely to pull out a heart? Explain your answer. What's the probability of picking the seven of diamonds from the stack of numbered cards? What's the probability of picking the king of clubs from the stack of face cards?"

Concerning the first example, we feel that all comment is superfluous, except to say that it can also be used in ludic and manipulatory activities and can be proposed even to children in the first two years of school. Up to the question "How many tickets did Michael buy?" the second example is an arithmetic problem. The need for analysis and determination of probabilities appears in the final two questions. In the third example, the most important aspect is represented by the fact that if the calculations are not performed it is impossible to discover that both choices lead to the same result.

5. Final observations

The ability to use the fraction symbol correctly in different contexts calls for a long-term teaching project which, through continuous feedback, must keep the teacher's work in tune with the child's developing psychological makeup. We agree with Bruner (1964) in saying that anything can be correctly and usefully translated into the way of thinking of a child of school age and that these first representations can be brought up later on and discussed more precisely and in greater detail. The idea of presenting the fraction as a single symbol with different meanings appears to us to be a feasible one as long as the conditions laid down are respected. The second approach involves fewer conditions, but it is no less important for this reason, since it is a question of including the analysis of probability, in the classic sense of the term, in the traditional concept of whole/part. In fact, the number of possible cases can be related to the "whole" while the number of most probable cases can be related to the number of parts taken into consideration. It is evident that the second approach is more feasible and is already in practice, though perhaps not always completely and knowledgeably, by teachers. The first approach is certainly less feasible owing to the long-term planning it entails, but it is also the one that carries with it a certain educative weight if we take into

account its interdisciplinary and multidisciplinary implications. The possibility of transferring aleatory and non-aleatory situations from one cultural context to another and the possibility of using appropriately the different linguistic syntagmas are a part of the necessary and natural development of cognitive processes. In fact, independently of the hypotheses proposed herein, we feel it is important to underline the fact that the inclusion of probability in primary school syllabuses constitutes for children a new and further opportunity for learning and developing cognitive capacities and gaining the ability to judge, all acquisitions that are essential to life in a society of increasing complexity (Caredda and Puxeddu, 1991b).

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