PROBABILISTIC THINKING VERSUS STATISTICAL THINKING

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We discuss the similarities and differences between probabilistic and statistical thinking, arguing in favour of metaphorising them as arrows pointing in opposite directions. We base our argument and discussion on concrete examples, drawn from our teaching experience to a broad spectrum of learners at the University of Chile and inspired by Brousseau's fundamental adidactic situation for inferential statistics in primary school, and our own fundamental adidactic situations for probability. Learners involved include first year humanistic university students, prospective math teachers, and in-service primary school teachers and their students.

INTRODUCTION

Nowadays it is quite frequent in our country, and elsewhere, to see mathematics teacher initial formations where probability appears subsumed in statistics; no explicit teaching of probability on its own survives. In our view this is as sensible as stopping teaching mathematics because it is enough to teach theoretical physics or mathematical physics, where mathematics is subsumed. Other common misconceptions in our view are that statistical thinking is just applied probabilistic thinking and that probabilistic thinking is just a special case of a measure of theoretical thinking (in the case of total mass 1). However, whilst probabilistic thinking is an (important) aspect of mathematical thinking, statistical thinking, like physical thinking, does not really pertain to mathematical thinking (Batanero & Borovcnik, 2016). This is well illustrated by the well-known joke: "in statistics 80/100 is not the same as 8/10, as in mathematics."

We deem probabilistic thinking and statistical thinking to be different kinds of thinking, which could be fittingly metaphorised as arrows pointing in opposite directions. We illustrate below our arrow metaphor in the case of Brousseau's well-known fundamental adidactic situation for inferential statistics in primary school (Brousseau et al., 2001), where students try to figure out the composition of an opaque bottle containing marbles of two different colours just by sampling the balls one at a time through a tiny window cut out in the cap of the bottle. Experimentally we observe that in fact both kinds of thinking usually emerge among the students working in random small groups.

In our view, these opposite arrows nevertheless *intertwine*, e.g., in the frequentist approach to probability, and *meet* at the Law of Large Numbers.

We point out *en passant* that the aforementioned intertwining has fostered some infelicitous terminology—ubiquitous in American textbooks—such as calling *the* "empirical probability" of an event its (relative) frequency in *a* given number of repetitions of the corresponding random experiment, as opposed to its "theoretical probability." A non-sensical terminology, because probabilities are theoretical by definition, probabilists from continental Europe would say (e.g., Humbenberger, 2019). In our country, we have even seen textbook authors devoting one chapter to relative frequencies and another to empirical probabilities!

We can also describe a theory by pointing out two characteristic or paradigmatic questions addressed by the theory, which trigger specific ways of thinking. In the case of probability and statistics the paradigmatic questions are "impossible questions" such as:

- How can we figure out the composition of a population from which we cannot make a census?

in the case of statistics, and

- Where will a random walker be after a given number of steps?

in the case of probability.

We develop these ideas below, based on concrete examples from our teaching to various kinds of students and learners, from an enactivist and metaphorical perspective.

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A FUNDAMENTAL ADIDACTIC SITUATION FOR STATISTICS: WHAT IS IN THE BOTTLE?

Brousseau's well-known fundamental adidactic situation for inferential statistics, first implemented with fourth graders in France, is designed as follows (Brousseau et al., 2002; Warfield, 2014):

The teacher takes at random, blindly, five marbles from a big bag containing as many black as white marbles and puts them in a small opaque bottle, which she closes hermetically. The bottle has, however, a small window in its cap that enables the students to peek one marble at a time. Then she hands the bottle over to a small group of students (up to five) and asks about the number of black and white marbles in it. She does the same for each one of the groups of students the class was partitioned into.

This situation is a first achievement in Brousseau's programme to reshape the teaching of mathematics in the French elementary school (Brousseau, 1965), which he found relentlessly abusive and punitive. His adidactic situations (Brousseau, 1998) and "leçons sans paroles" (Brousseau, 1965) inscribed in an "experimental epistemology of mathematics" may be seen as forerunners of enactivist situations in the learning of mathematics (and statistics).

We should mention here that since 2010, in contrast with the French curriculum, the Chilean curriculum emphasises COPISI (an acronym for COncreto–PIctórico–SImbólico), analogue to Singapour's CPA (Concrete–Pictorial–Abstract) or the German EIS Prynzip (Enaktiv–Ikonisch–Symbolisch), which inspired their Chilean counterpart.

In this adidactic situation the students work on their own on the problem, while the teacher steps back most of the time, just asking a few questions and observing how students self organise. Here statistical thinking emerges when the students—working in small groups—begin to explore different ways of organising their sampling: they could decide to observe the bottle by series of five, for instance, and choose the most frequent composition after many repetitions, or just sample one by one and look at the proportion of black marbles observed.

French students (9–11-year-olds) usually have first the idea of observing the marbles in series of five and look for the most frequent composition; only later they think of *accumulated frequencies* (contrary to what we have observed in older Chilean students, prospective teachers included).

It is pertinent to remark that not infrequently the *mode* of the series of five says one thing and the accumulated frequencies say another thing. Several of our working groups have met this phenomenon with 20 series of five and 100 single marble observations, to be more precise. When students express their bewilderment, we suggest that they look for an explanation of this surprising phenomenon in an enactive way, handling very *concrete* objects (coins or tokens, for instance). A series of observations is not so concrete after all …

The learners discuss how many samplings "are enough" to be able to "guess" the right composition of the bottle. Eventually they realise that there is not a magic number of observations (such as 666 or so) but that this is a matter of behaviour and tendency, which becomes quite visible when they plot, say, the *accumulated frequency* of black in n observations as a function of n.

One important feature of this fundamental situation is that the opaque bottle cannot, and must not, be opened. The teacher resists stubbornly the requests of the students, who want to open the bottle "to check whether they were right." In fact, when the students are convinced that they have figured out the composition of the bottle, the teacher empties the bottle back to the big bag quickly so that nobody sees the colours of the marbles inside. Of course, in real life statistics, you cannot drain the whole lake to figure out the composition of its fish population or have every citizen undergo an AIDS screening test or a PCR test (unless you are in China).

At some point, they wonder whether a black–to–white ratio of 3:2 for the bottle could obnoxiously behave for a long time like a 2:3 bottle. Eventually, one student suggests *creating* a transparent 3:2 bottle and observe how it behaves under sampling, a *probabilistic insight* that entails a cognitive reversal, we claim. So actually, in this fundamental situation for inferential statistics, probabilistic insights and thinking usually emerge at some point.

From our perspective, what is typically probabilistic thinking here is the attempt to predict the behaviour of a given transparent bottle under repeated sampling. What is statistical thinking is the attempt to infer the composition of a given opaque bottle after the results of repeated sampling. Both types of thinking attempt to give sensible answers to "impossible questions":

- What is the composition of a population from which we cannot make a census?
- Where will a random jumping frog be after a given number of jumps?

Incidentally, we also prompt the students to look for a more visual or geometric way to represent the *sampling process* they are engaged in. Those who have played around a bit with random walks before (see next section) think quite easily of metaphorising their sampling process as a *random walk* in the plane: one step right if the observed marble is white, one step upwards if it is black, say. They see that their random walk, after some initial wild fluctuations, ends up confined in a fixed wedge in the plane. Notice that the slope of the line connecting the origin with the current position of the walker (the global linear interpolation of the walk) gives the current black–to–white ratio.

OUR FUNDAMENTAL SITUATION FOR PROBABILITY: WHERE IS THE FROG?

Our underlying idea is to set up a fundamental situation based on a *stochastic process*, typically a *random walk*, where probabilities emerge idiosyncratically metaphorised as *portions* of the walker. For example: *A frog jumps happily and symmetrically (as if tossing a coin to decide, between right and left) on a row or a regular polygon of stones in a pond* (Soto-Andrade et al., 2018). At primary school, fourth or fifth grade, the frog could jump symmetrically just on a *row of nine stones* in a pond, simulated with the help of a coin, or a dice (easier to toss). The fundamental but awkward question emerges: *Where will the frog be after one, two, three, four jumps, if it starts from the central stone?*

Learners may enact the frog themselves or simulate it, flipping a coin to choose right or left. They can bet on some stone where the frog would land after two or three jumps, for instance. A collective simulation may be carried out, where every student in the class would simulate a frog and then a snapshot would be taken of the final total frog distribution on the stones. Thanks to a betting game, most of them will learn the hard way that not all stones are equally likely (8-year-olds seem to realise this more easily than 6-year-olds). Students in fourth or fifth grade can represent data obtained through simulation and, from there, propose an "ideal" distribution of frogs after a given number of jumps.

Notice that the simplest polygonal random walk occurs in a triangle of stones, where the frog jumps from each stone to one of the two other stones, equally likely.

We claim that the "impossible question" about the future of the floor (including its fate at "the end of times") is a typical probabilistic question. It can be tackled statistically, though, by simulating the frog's random walk "many times" and registering the frequency of the frog squatting at a given stone after a fixed number of jumps. Notice that experimentally, the students may arrive to the conclusion that the frog tends to become equally distributed (a kind of quantum frog) in the long run, but they have some trouble giving a sensible quantitative answer to the impossible question for a given number of jumps. Probabilistic thinking emerges, in our view, as an alternative to statistical simulation when the students try to predict theoretically the fate of the random walker.

An interesting variant of this is the 2D random walk of a puppy in a city, as described in the following story:

> *A little puppy, fittingly called Brownie, escapes randomly from home, when she smells the shampoo her master intends to give her. At each street corner, confused by the traffic's noise and smells and escaping barely from being overrun, she chooses equally likely any of the 4 cardinal directions and runs nonstop a whole block until the next corner. Exhausted, after 4 blocks, say, she lies at some corner. Her master would like to know where to look for Brownie and also to estimate how far she will end up from home.*

Working in groups, the learners realize quickly that there are impossible corners (street crossings) for Brownie, even close to her home. Quite often, after having spotted the corners where Brownie could possibly be (after a four-block run), several (up to one half of the) students or teachers will deem them equally likely. Others have the vague intuition that some corners are more likely because Brownie can get there by several different paths.

In secondary school in Chile the idea of experimenting by simulating a big number of puppies (statistical thinking) arises rarely. Simulation, however, helps the majority of students to become convinced that some corners are more favourable to find Brownie—her home, for instance. They have little trouble in setting up a corner *ranking.* But they struggle to quantify their feeling of bigger or smaller likelihood. How to assign "weights" to the different corners? Thinking statistically, they would observe the corresponding frequencies for increasing numbers of simulations of the four-block walk. Some think of counting paths to quantify: an example of probabilistic thinking à la Laplace. They find 256 possible paths for a four-block random walk, though, from which the favourable ones need to be extracted!

However, there are friendlier and more efficient *metaphorical avatars of probabilistic thinking*, we claim. In this case, under some minimal prompting such as: *What else, more concretely, could you imagine instead of this (rather abstract) fair random choice between the four cardinal directions?* Using some discrete gesture language, metaphors emerge idiosyncratically, such as the ones we have dubbed:

- "Solomonic metaphor" (cut the puppy into four pieces and so on) or
- "Hydraulic metaphor" (the puppy flows, like water, equitably to the four immediate neighbours and so on).

Typically, in a class of 30, one or two immediately "see" the puppy splitting into four pieces. Those who feel ill at easy doing that, suggest eventually replacing the puppy by a piece of chocolate or a litre of fruit juice. Often in-service teachers seem to have the feeling that they are violating some (unspoken) didactical contract when allowing themselves to metaphorise. The learners usually do not have much trouble quantifying the likelihood of presence of the puppy at the different possible corners, after a given number of steps, with the help of the hydraulic metaphor. They realise quickly the "conservation law of the puppy": putting together her pieces at each step you reconstruct the whole puppy.

Another metaphor for the random walk might also emerge in the classroom: unleashing a pack of puppies from home, that would spread out evenly, splitting into four equal groups at each corner (the students begin with 16 puppies that will run 2 blocks each). We call this a "pedestrian metaphor." Notice that this sort of metaphorical probabilistic thinking may be easily enacted. For instance, for the 2D two-block random walk of Brownie, 9 + 4 = 13 students distribute themselves suitably on the nodes of a grid drawn on the floor, as shown in Figures 1 and 2 (Soto-Andrade, 2013), the central one holding a container with one litre of water that she shares equitably with her immediate four neighbours (see Figure 1). Then each of these four students does the same and they end up with the water distribution shown in Figure 2. In this way, each one of them can find out easily the amount of water that they will have after a given number of steps. Of course, they can also simulate this (more easily although less dramatically) with a square board that they partition into four pieces and so on. However, in this way they will have to figure out sums of bunches of fractions of the original cardboard …

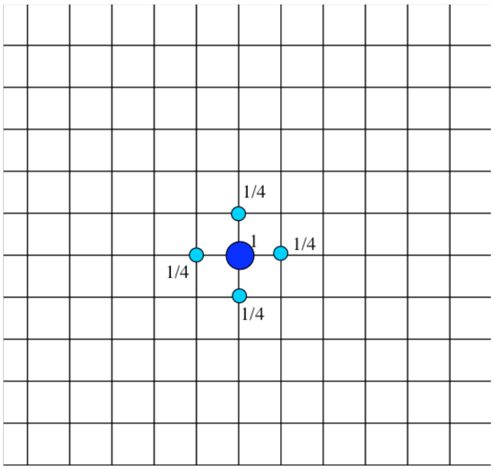

Figure 1. Brownie's splitting (1 step)

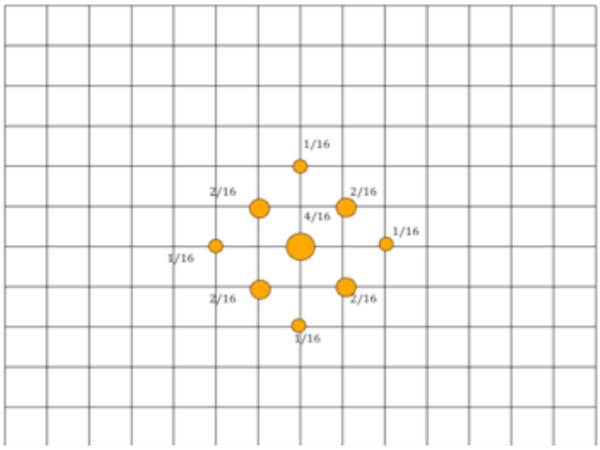

Figure 2. Brownie's splitting (2 steps)

According to our teaching experience, most students and in-service teachers succeeded in this way in beginning to think probabilistically, constructing the concept of probability at the same time that they solved this sort of problem in context, with the help of suitable metaphorising (Soto-Andrade et al., 2018; Soto-Andrade, 2013, 2021).

In our view, indeed, probabilistic thinking is not just drawing possibility trees with probabilistic weights calculated by the multiplication rule; it is much more constructing the notion of probability by a metaphorical sleight of hand, as portions of a random walker who splits instead of choosing randomly one direction or another.

CONCLUSION

We have commented two paradigmatic examples of fundamental adidactic situations (Brousseau, 1998) that work as triggers of statistical thinking and probabilistic thinking.

The first one (the opaque bottle) is intended as a fundamental situation for (inferential) statistical thinking. Interestingly, probabilistic thinking ("let us make up a transparent bottle and see what happens when we sample") emerges idiosyncratically in almost every class. In this example, we clearly see statistical and probabilistic thinking as "arrows pointing in opposite directions." For instance, the former tries to figure out the composition of the bottle from the sampling data, the later attempts to predict the sampling data from the composition of the bottle. Our metaphor also admits a chronological interpretation: statistical thinking takes support in collected, past data, and probabilistic thinking eyes future data. So, in a sense, statistics looks at the past, and probability looks at the future (Brousseau et al., 2002, p. 408). On the other hand, both are observer–dependent and try to provide sensible answers to "impossible questions" for which there are no sure or definite answers (as in classical mathematical thinking).

The second one (the frog's 1D random walk or the puppy's 2D random walk) is intended as a fundamental situation for probabilistic thinking, either in its classical Laplacian avatar (counting possible paths, as equally likely outcomes) or in its—friendlier and more efficient—metaphorical avatar (splitting the walker or "hallucinating" an iterated sharing process instead). In this case, we also observe statistical thinking, when the learners get the idea of simulating many times the random walk, to try to fathom how much more likely is one corner than another.

Finally, we may notice interesting connections between both situations. The sampling process in the first situation may be seen as a random walk in the plane (one step right if white, one step up if black, say). The "average slope" of this random walk appears to tend to the black/white marble ratio.

All possible paths for the four-block random walk in the second situation may be seen as balls in an urn, where, for instance, all paths leading back to the Brownie's home are black balls, say, and all other paths are white balls (or other balls of different colours according to the corner to which they lead). Then, simulating the four-block random once amounts to sampling one ball from the urn, whose composition is now known. An analogous case to the opaque bottle situation would be the one where the transition probabilities of the random walk are unknown, but we can simulate it to empirically sample the location of the walker as many times as we wish, after four blocks, say.

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